Superdiffusion

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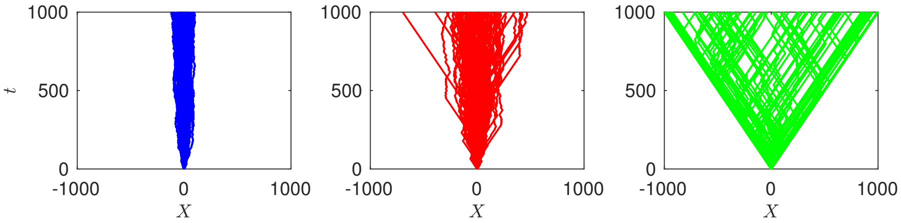
Abstract. Think about a cloud of particles originally concentrated in a single location, which we'll call the origin. If the particles move in random directions, but all with the same constant speed, there will be a *wave front* that will be at a distance $\sim t$ from the origin at time t. If the particles interact with a background — think of a pollutant in air — and frequently change directions as a result, the cloud will *diffuse*. Its "size" $\sqrt{E(\|X(t)\|^2)}$, where $X(t)$ = particle location at time t, will be $\sim \sqrt{t}$.

In recent decades, it has been realized that in between these two extremes, which one might call "hyperbolic" and "parabolic", lies a large range called *superdiffusive*, in which the expansion is asymptotically faster than $\sqrt{t}$, but slower than t. Superdiffusion arises in situations in which segments of long unimpeded motion are rare, but not without consequence. Such behavior arises commonly in Physics, Biology, as well as in pure dynamical systems.

The following plot shows space-time trajectories of random walkers on the real line in the parabolic (left), superdiffusive (center), and hyperbolic or "ballistic" (right) regimes.



We ask exactly how fast the cloud expands in the superdiffusive regime. Using nothing beyond Calculus, elementary Probability, and a very small amount of Analysis, we re-derive answers to this question that had previously been obtained in the Physics literature using less elementary approaches, and prove extensions that appear to be new.

This is joint work with Claude Greengard.