Structured additive regression for multicategorical space-time data: a mixed model approach

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Forest health data

- Data collected in yearly forest health inventories carried out in a forest in northern Bavaria from 1983 to 2001.

- 83 observation points with beeches in an area extending 15 km from east to west and 10 km from north to south.

- y_{it}, the defoliation degree of beech i in year t, is measured in three ordered categories (multicategorical response):

  $y_{it} = 1$ no defoliation,
  $y_{it} = 2$ defoliation 25% or less,
  $y_{it} = 3$ defoliation above 25%.

- Covariates:

  t calendar time,
  s_i site of the beech,
  a_{it} age of the tree in years,
  u_{it} further (mostly categorical) covariates.

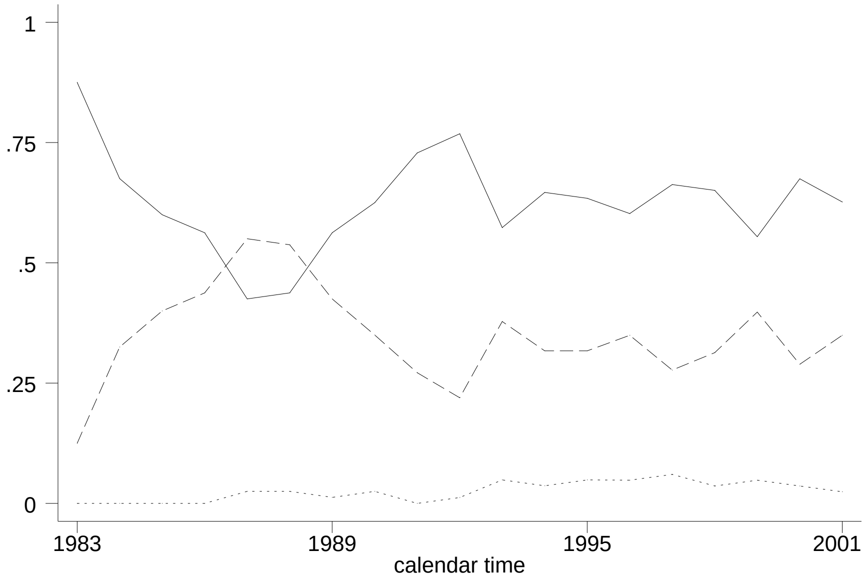

Figure 1: Temporal development of the frequency of the damage states:

— no damage,

- - - medium damage,

· · · severe damage.

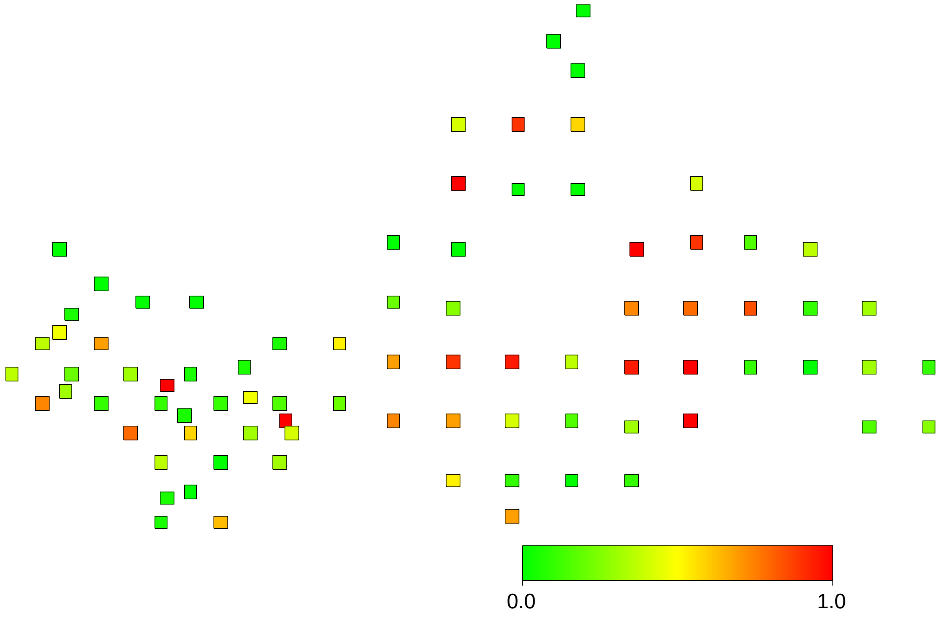

Figure 2: Spatial distribution of the beeches and percentage of time points for which a beech was classified to be damaged (damage state 2 or 3).

Regression models for ordinal responses

- Response y_{it} follows multinomial distribution with three ordered categories $r = 1, 2, 3$.

- Model the cumulative probabilities

$$P(y_{it} \leq r) = F(\theta_r - \eta_{it})$$

with thresholds $-\infty = \theta_0 < \theta_1 < \theta_2 < \theta_3 = \infty$ and linear predictor η_{it}.

- $F(\cdot)$ can be any cumulative distribution function:

standard normal $\Longrightarrow$ cumulative probit model,
logistic $\Longrightarrow$ cumulative logit model.

- Consider a random variable with density $f = F'$ and expectation η_{it}.

  $\Longrightarrow$ Linear predictor determines shift on latent scale.

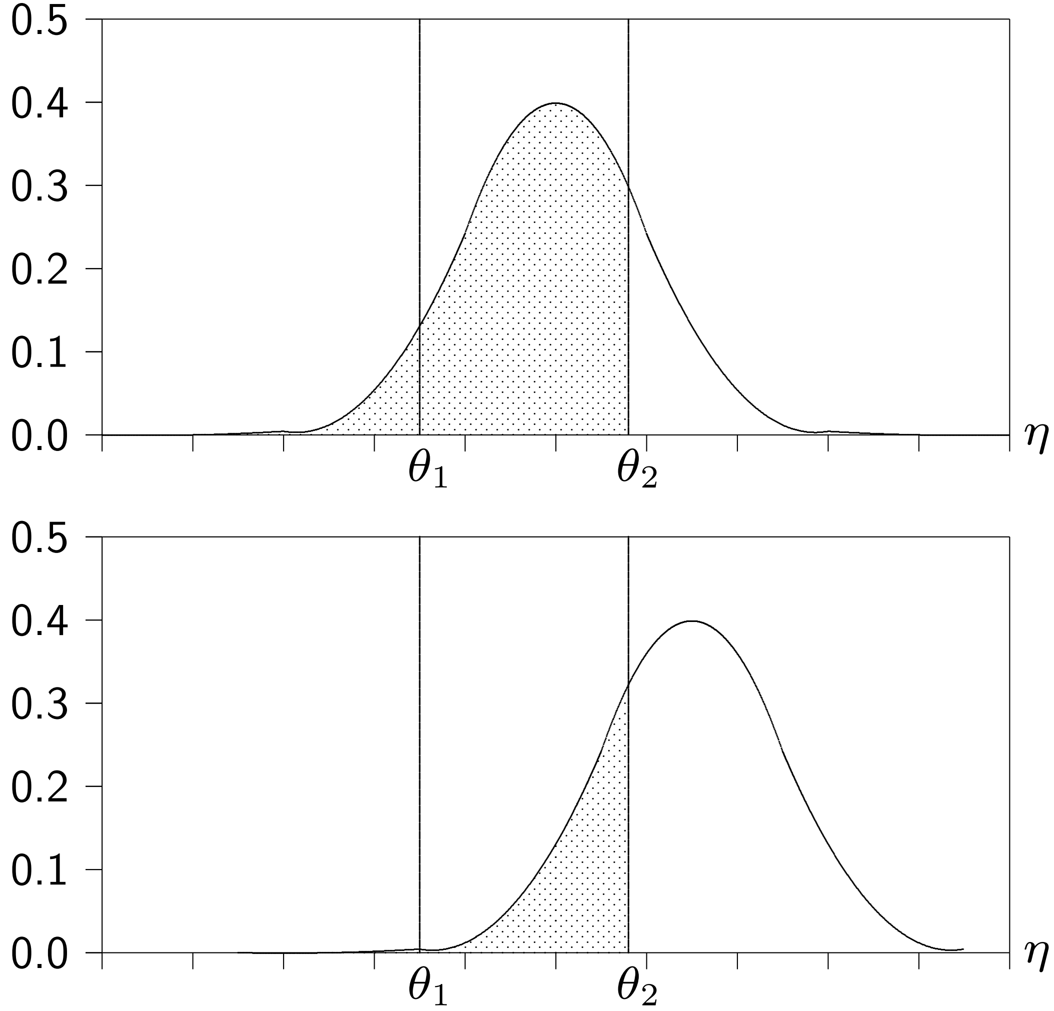

Figure 3: The shaded areas represent $P(y_{it} \leq 2)$ for different values of η_{it}.

- Limitations of a purely parametric approach:
  - Spatio-temporal structure of the data implies spatial and temporal correlations.
  - Nonlinear effects of continuous covariates?
  - Complex interactions between covariates?

$\Longrightarrow$ Structured additive regression models.

Structured additive regression

- Replace usual parametric predictor with a flexible semiparametric predictor

$$\eta_{it} = f_1(t) + f_2(a_{it}) + f_3(t, a_{it}) + f_{spat}(s_i) + u_{it}'\gamma,$$

  where

  - f_1 and f_2 are nonparametric functions of calendar time and age,
  - f_3 is an interaction surface between calendar time and age,
  - f_{spat} is a spatial function, and
  - u is a vector of further covariates with parametric effects.

- Structured additive regression extends (and combines) generalized additive mixed models, geoadditive models and varying coefficient models.

- Allows unified treatment of all effects within a Bayesian framework.

- $f_1(t)$, $f_2(a_{it})$: P-splines
  - Approximate f_j by a B-spline of a certain degree (basis function approach).
  - Penalize differences between parameters of adjacent basis functions to ensure smoothness.
  - Alternatives: Random walks, more general autoregressive priors.
- $f_3(t, a_{it})$: Two-dimensional extensions of P-splines
  - Define two-dimensional basis functions based on tensor products of one-dimensional B-splines.
  - Use priors from spatial statistics for penalization.
  - Alternative: Varying coefficient models, if one of the interacting variables is categorical.

- $f_{spat}(s_i)$: Markov random fields
  - Consider two trees as neighbors if their distance is less than (e.g.) 1.2 km.
  - Assume that the expected value of $f_{spat}(s)$ is the average of the function evaluations of adjacent sites.
- $f_{spat}(s_i)$: Stationary Gaussian random fields (kriging)
  - Spatial effect follows a zero mean stationary Gaussian stochastic process.
  - Correlation of two arbitrary sites is defined by an intrinsic correlation function.
- Split up spatial effect into structured and unstructured part:

$$f_{spat}(s_i) = f_{str}(s_i) + f_{unstr}(s_i)$$

The unstructured effect can be modelled by i.i.d. random effects, the structured effect by a MRF or a GRF.

- All effects f_j can be expressed as the product of a design matrix Z_j and a vector of regression coefficients β_j.

- Rewrite the structured additive predictor in matrix notation as

$$\eta = Z_1\beta_1 + Z_2\beta_2 + Z_3\beta_3 + Z_{spat}\beta_{spat} + U\gamma.$$

- Bayesian approach: Assign an appropriate prior to β_j.

- All priors can be cast into the general form

$$p(\beta_j|\tau_j^2) \propto \exp\left(-\frac{1}{2\tau_j^2}\beta_j' K_j \beta_j\right)$$

  where K_j is a penalty matrix and τ_j^2 is a smoothing parameter.

- Type of the covariate and prior beliefs about the smoothness of f_j determine special Z_j and K_j.

Mixed model representation

- Each parameter vector β_j can be partitioned into an unpenalized part (with flat prior) and a penalized part (with i.i.d. Gaussian prior) yielding a variance components model

  $$\eta = X^{unp}\beta^{unp} + X^{pen}\beta^{pen}$$

  with

  $$p(\beta^{unp}) \propto const \qquad \beta^{pen} \sim N(0, \Lambda)$$

  and

  $$\Lambda = \text{blockdiag}(\tau_1^2 I, \dots, \tau_4^2 I).$$

- Regression coefficients are estimated via modified Fisher scoring.

- The mixed model representation allows for restricted maximum likelihood / marginal likelihood estimation of the variance components:

$$L(\Lambda) = \int L(\beta^{unp}, \beta^{pen}, \Lambda) p(\beta^{pen}) d\beta^{pen} d\beta^{unp} \rightarrow \max_{\Lambda}.$$

- From a Bayesian perspective, we get empirical Bayes / posterior mode estimates.
- Closely related to penalized likelihood.
- Fahrmeir, Kneib and Lang (2004) derive numerically efficient formulae that allow the computation even for fairly large data sets.

Results

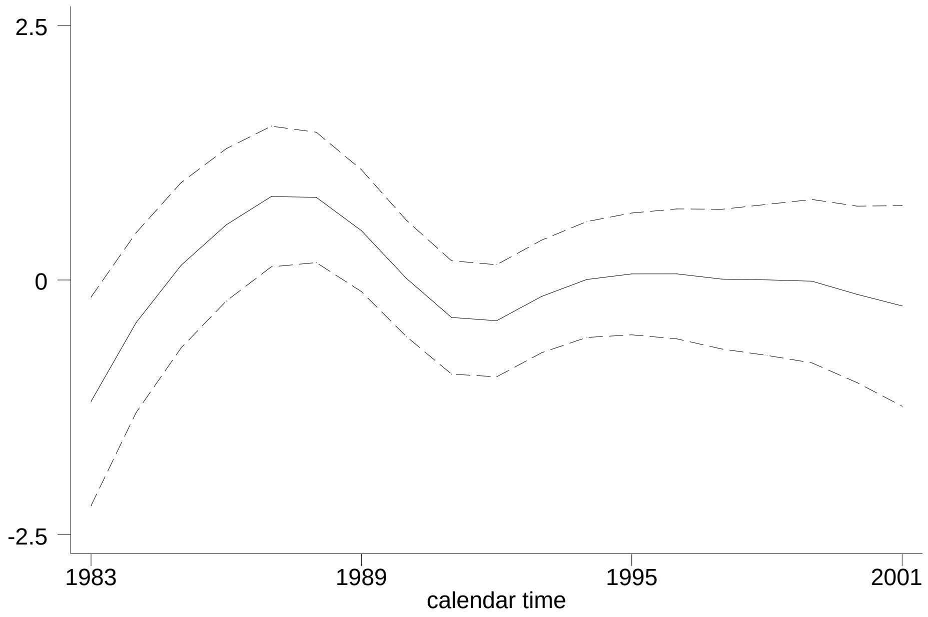

Figure 4: Time trend.

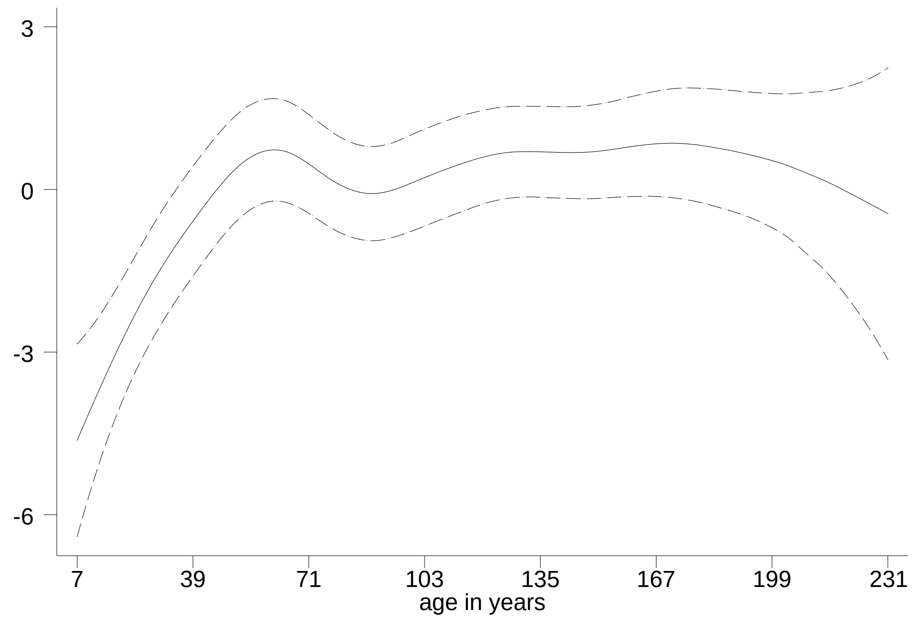

Figure 5: Age effect.

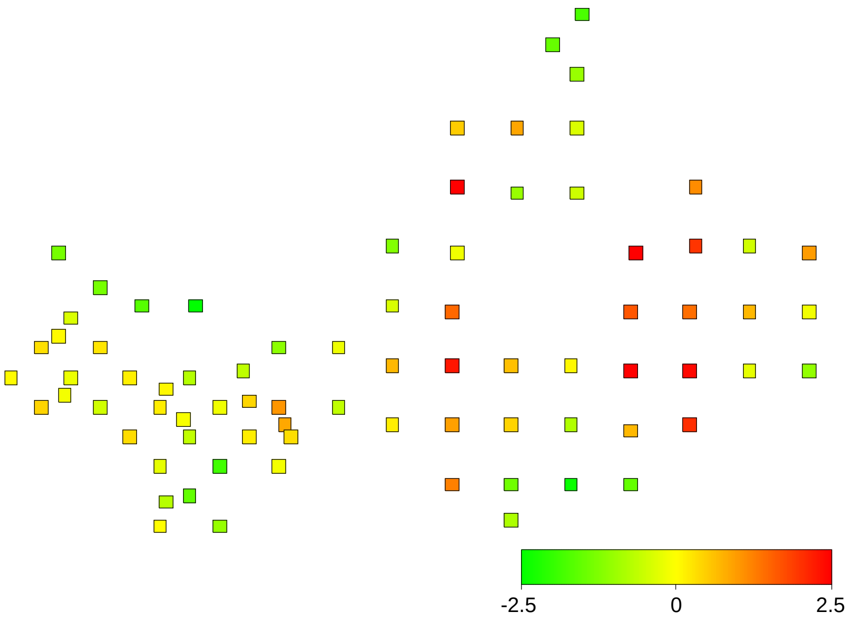

Figure 6: Structured spatial effect.

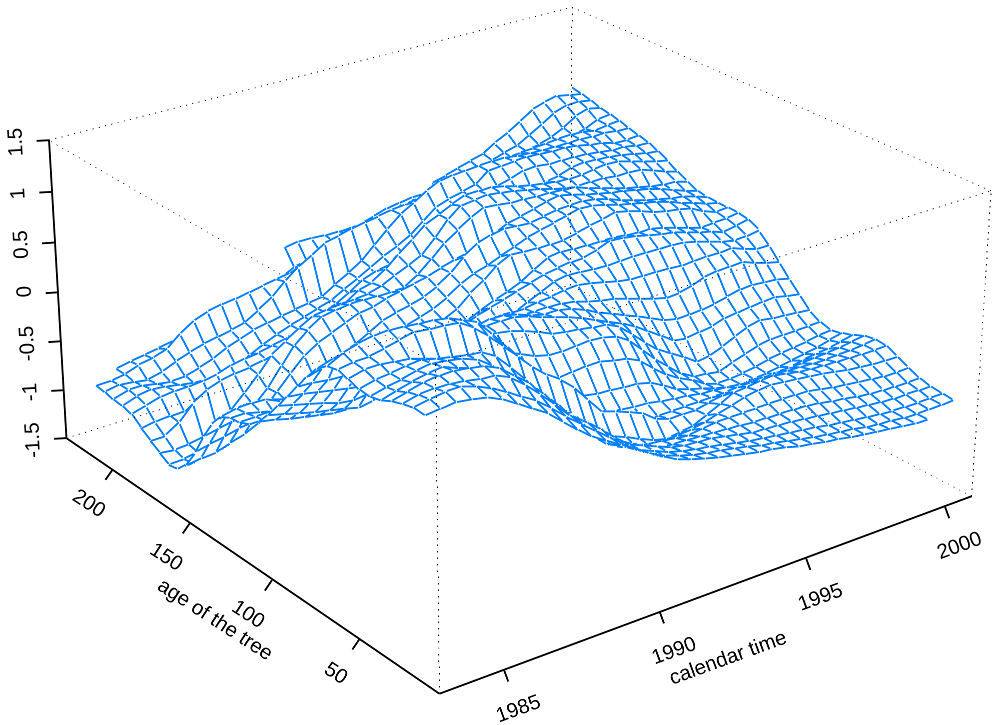

Figure 7: Interaction effect.

Software

- Estimation was carried out using BayesX, a public domain software package for Bayesian inference.

- Available from

`http://www.stat.uni-muenchen.de/~lang/bayesx`

- Features (within a mixed model setting):
  - Responses: Gaussian, Gamma, Poisson, Binomial, ordered and unordered multinomial.
  - Continuous covariates and time scales: Random Walks, P-splines, autoregressive priors for seasonal components.
  - Spatial Covariates: Markov random fields, stationary Gaussian random fields, two-dimensional P-Splines.
  - Interactions: Two-dimensional P-splines, varying coefficient models with continuous and spatial effect modifiers.
  - Random intercepts and random slopes.

Discussion

- Models for nominal responses are supported, too.
- Comparison with fully Bayesian approach based on MCMC:

  Cons:

  - Credible intervals rely on asymptotic normality.
  - Only plug-in estimates for functionals.

  Pros:

  - No questions concerning mixing and convergence.
  - No sensitivity with respect to prior assumptions on variance parameters.
  - Somewhat better point estimates (in simulations).

- Future work:
  - Multinomial probit models with correlated latent utilities.
  - Category-specific covariates in nominal models.
  - Covariate-dependent thresholds in ordinal models.

References

- Fahrmeir, L., Kneib, T. and Lang, S. (2004): Penalized structured additive regression for space-time data: A Bayesian perspective. Statistica Sinica (to appear).

- Kneib, T. and Fahrmeir, L. (2004): Structured additive regression for multicategorical space-time data: A mixed model approach. SFB 386 Discussion Paper 377, University of Munich.

- Both available from

`http://www.stat.uni-muenchen.de/~kneib`