Introduction to Spectral Theory

First lecture: Bounded operators

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Summer School on Operator Algebras, Spectral Theory and Applications to Topological Insulators

Tbilisi State University

September 17-21, 2018

Linear operators between Hilbert spaces

All Hilbert spaces considered in these lectures will be over $\mathbb{C}$ and be separable.

Let $\mathcal{H}$, $\mathcal{H}'$ be Hilbert spaces.

Lemma

For a linear operator $A\colon \mathcal{H} \to \mathcal{H}'$, the following conditions are equivalent:

- *A is continuous,*
- *A is bounded, i.e., $\|Ax\| \leq C\|x\|$ for some $C \geq 0$,*
- graph $A = \{(x, Ax) \mid x \in \mathcal{H}\} \subset \mathcal{H} \times \mathcal{H}'$ *is closed* (closed graph theorem).

Remark The best constant C is $\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y\rangle|$.

We write $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ for the space of these linear operators A and $\mathcal{L}(\mathcal{H})$ in case $\mathcal{H} = \mathcal{H}'$.

$\mathcal{L}(\mathcal{H})$ as a C^*-algebra

$\mathcal{L}(\mathcal{H})$ equipped with the operator norm $\| \ \|$ is a Banach algebra (in particular, $\|AB\| \leq \|A\|\|B\|$ for A, $B \in \mathcal{L}(\mathcal{H})$).

Recall that the adjoint $A^* \in \mathcal{L}(\mathcal{H})$ of $A \in \mathcal{L}(\mathcal{H})$ is defined by

$$\langle Ax, y \rangle = \langle x, A^* y \rangle, \quad x, y \in \mathcal{H}.$$

With the involution $A \mapsto A^*$, $\mathcal{L}(\mathcal{H})$ is in fact a C^*-algebra (in particular, $\|A^* A\| = \|A\|^2$ for $A \in \mathcal{L}(\mathcal{H})$).

Objective of these lectures Understand the spectral theory of self-adjoint operators $A \in \mathcal{L}(\mathcal{H})$.

Remark One could equally well study the spectral theory of self-adjoint elements of an abstract unital C^*-algebra.

Topologies on $\mathcal{L}(\mathcal{H}, \mathcal{H}')$

There are three natural topologies on $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ of decreasing strength.

We define convergence of a sequence $\{A_n\} \subset \mathcal{L}(\mathcal{H}, \mathcal{H}')$ for each of these topologies:

1. $A_n \to A$ if $\|A - A_n\| \to 0$ (uniform operator or norm topology),
2. $A_n \xrightarrow{s} A$ if $A_n x \to Ax$ for each $x \in \mathcal{H}$ (strong operator topology),
3. $A_n \xrightarrow{w} A$ if $\langle A_n x, y\rangle \to \langle Ax, y\rangle$ for each $x, y \in \mathcal{H}$ (weak operator topology).

Theorem (Uniform boundedness principle)

Let $\{A_n\} \subset \mathcal{L}(\mathcal{H}, \mathcal{H}')$ be a sequence. Suppose that $\{\langle A_n x, y\rangle\} \subset \mathbb{C}$ converges for all $x \in \mathcal{H}$, $y \in \mathcal{H}'$. Then $A_n \xrightarrow{w} A$ for some $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$.

Resolvent and spectrum

Let $A \in \mathcal{L}(\mathcal{H})$.

Definition

1. $\lambda \in \mathbb{C}$ belongs to the resolvent set $\rho(A)$ if $(A - \lambda)^{-1} \in \mathcal{L}(\mathcal{H})$.
2. The resolvent is $R(\lambda, A) = (A - \lambda)^{-1}$ for $\lambda \in \rho(A)$.
3. The spectrum is $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

Theorem

$\sigma(A)$ is a non-empty, compact subset of $\mathbb{C}$ contained in $B(0, \|A\|) = \{\lambda \in \mathbb{C} \mid |\lambda| \le \|A\|\}$.
The map $\rho(A) \to \mathcal{L}(\mathcal{H})$, $\lambda \mapsto R(\lambda, A)$ is holomorphic.

Proof For $\lambda > \|A\|$, $R(\lambda, A) = -\sum_{j=0}^{\infty} \lambda^{-(j+1)} A^j$.
For $\lambda \in \rho(A)$, $|\mu - \lambda| < \|R(\lambda, A)\|^{-1}$, $R(\mu, A) = \sum_{j=0}^{\infty} (\mu - \lambda)^j R(\lambda, A)^{j+1}$.
Finally, invoke Liouville's theorem to conclude that $\sigma(A) \neq \emptyset$. $\square$

Spectral radius

The spectral radius is defined as $r(A) = \sup_{\lambda \in \sigma(A)} |\lambda|$.

Proposition

Let $A \in \mathcal{L}(\mathcal{H})$. Then

(a) $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$,

(b) $r(A) = \|A\|$ *if A is normal.*

Discrete and essential spectrum

- $\lambda \in \mathbb{C}$ is an eigenvalue of A if $A - \lambda$ is not injective.
- $\ker(A - \lambda)$ is called the eigenspace of A belonging to the eigenvalue λ, a non-zero element u of $\ker(A - \lambda)$ (i.e., $u \neq 0$ and $Au = \lambda u$) is called an eigenvector.

The discrete spectrum $\sigma_{\mathrm{d}}(A) \subseteq \sigma(A)$ consists of isolated eigenvalues of A of finite multiplicity (i.e., $\ker(A - \lambda)^N = \ker(A - \lambda)^{N+1}$ for some $N \in \mathbb{N}$ and $\dim \ker(A - \lambda)^N < \infty$).

Remark Eigenvalues of normal operators are semi-simple (i.e., $N = 1$).

The essential spectrum is defined as $\sigma_{\mathrm{e}}(A) = \sigma(A) \setminus \sigma_{\mathrm{d}}(A)$.

Multiplication operators, I

Let (X, μ) be a measure space. Then each $g \in L^\infty(X, \mu)$ induces a bounded operator

$$M_g \colon L^2(X, \mu) \to L^2(X, \mu), \quad u \mapsto g \cdot u$$

(multiplication operator).

The essential range $\operatorname{ran} g$ consists of all $\lambda \in \mathbb{C}$ such that, for all $\epsilon > 0$,

$$\mu\left(\{|g - \lambda| < \epsilon\}\right) > 0.$$

Lemma

(a) $\sigma(M_g) = \operatorname{ran} g$.

(b) *λ is an eigenvalue of M_g if and only if $\mu\left(\{g = \lambda\}\right) > 0$.*

Classes of linear operators

Definition

An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be normal if A and A^* commute, i.e., $A^*A = AA^*$. Special cases are

1. (Self-adjoint operators) $A = A^*$,
2. (Unitary operators) $A^{-1} = A^*$.

Lemma

Let $A \in \mathcal{L}(\mathcal{H})$ be normal. Then

1. *A is self-adjoint if and only if $\sigma(A) \subseteq \mathbb{R}$,*
2. *A is unitary if and only if $\sigma(A) \subseteq \mathbb{S}^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.*

Remark One also has unitary operators $U \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ given by $U^*U = \mathrm{I}_{\mathcal{H}}$ and $UU^* = \mathrm{I}_{\mathcal{H}'}$ between different Hilbert spaces. Spectral properties do not change under unitary equivalence, i.e., under the map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$, $A \mapsto UAU^*$.

Multiplication operators, II

- Because of $(M_g)^* = M_{\bar{g}}$ and

$$M_{\bar{g}} M_g = M_g M_{\bar{g}} = M_{|g|^2},$$

  multiplication operators are normal.

- M_g is self-adjoint if and only if g is (essentially) real-valued, i.e., $\operatorname{ran} g \subseteq \mathbb{R}$.

Projections

Definition

$P \in \mathcal{L}(\mathcal{H})$ is said to be an orthogonal projection if $P = P^* = P^2$.

The complementary projection is $P^\perp = \mathrm{I} - P$. P projects onto $\operatorname{ran} P$ and we have

$$\mathcal{H} = \operatorname{ran} P \oplus \operatorname{ran}(\mathrm{I} - P),$$

where $\ker P = \operatorname{ran}(\mathrm{I} - P)$ and $\ker(\mathrm{I} - P) = \operatorname{ran} P$.

There is a (complete) lattice structure on the set of orthogonal projections given by $P \le Q$ if $\operatorname{ran} P \subseteq \operatorname{ran} Q$ (equivalently, $P = PQ = QP$). Moreover, we call two projections P, Q orthogonal (and write $P \perp Q$) if $P \le Q^\perp$ (equivalently, $PQ = QP = 0$).

Partial isometries

- $U \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ is said to be an isometry if $\|Ux\| = \|x\|$ for all $x \in \mathcal{H}$.
- $U \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ is said to be a partial isometry if it is an isometry when restricted to $(\ker U)^\perp$.
- $U \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ is a partial isometry if and only if U^*U and UU^* are projections. In this case, U maps unitarily from its initial space $(\ker U)^\perp = \operatorname{ran}(U^*U)$ onto its final space $\operatorname{ran} U = \operatorname{ran}(UU^*)$.

More on the essential spectrum

Theorem (Weyl's criterion)

Let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint, $\lambda \in \mathbb{R}$. Then

(a) $\lambda \in \sigma(A)$ *if and only if there is a sequence* $\{\varphi_n\} \subset \mathcal{H}$ *such that* $\|\varphi_n\| = 1$ *for all n and* $(A - \lambda)\varphi_n \to 0$ *in* $\mathcal{H}$*,*

(b) $\lambda \in \sigma_{\mathrm{e}}(A)$ *if and only if the sequence* $\{\varphi_n\}$ *in* (a) *can be chosen to be orthogonal* (*equivalently,* $\varphi_n \xrightarrow{\mathrm{w}} 0$)*.*

Theorem (Weyl)

Let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint and $B \in \mathcal{L}(\mathcal{H})$ be compact. Then

$$\sigma_{\mathrm{e}}(A) = \sigma_{\mathrm{e}}(A + B).$$

Compact operators, I

Lemma

For $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$, the following conditions are equivalent:

(a) $\overline{AB_1(0)} \subset \mathcal{H}'$ *is compact.*

(b) *A takes bounded sets in $\mathcal{H}$ to relatively compact sets in $\mathcal{H}'$.*

(c) *A takes weakly convergent sequences in $\mathcal{H}$ to strongly convergent sequences in $\mathcal{H}'$.*

In this case, A is said to be a compact operator. The set of all compact operators will be denoted by $\mathcal{K}(\mathcal{H}, \mathcal{H}')$ and by $\mathcal{K}(\mathcal{H})$ in case $\mathcal{H} = \mathcal{H}'$.

Compact operators, II

Examples

(a) Finite-rank operators are compact.

(b) The identity $\mathrm{I}_{\mathcal{H}} \in \mathcal{L}(\mathcal{H})$ is compact if and only if $\dim \mathcal{H} < \infty$.

Proposition

(a) *$\mathcal{K}(\mathcal{H}, \mathcal{H}')$ is norm closed in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$.*

(b) *If $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$, $K \in \mathcal{K}(\mathcal{H}', \mathcal{H}'')$, and $B \in \mathcal{L}(\mathcal{H}', \mathcal{H}'')$, then $BKA \in \mathcal{K}(\mathcal{H}, \mathcal{H}'')$.*

(c) *Every compact operator is the norm limit of a sequence of finite-rank operators.*

In particular, $\mathcal{K}(\mathcal{H})$ is a closed two-sided ideal in $\mathcal{L}(\mathcal{H})$.

Riesz-Schauder theory

Theorem

Let $K \in \mathcal{K}(\mathcal{H})$. Then $\sigma(K) \setminus \{0\}$ consists of isolated eigenvalues of finite multiplicity.

Corollary

Let $K \in \mathcal{K}(\mathcal{H})$ be self-adjoint. Then $\mathcal{H}$ possesses an orthonormal basis $\{\varphi_n\}$ consisting of eigenvectors of K, i.e., $K\varphi_n = \lambda_n \varphi_n$ for each n and some $\lambda_n \in \mathbb{R}$. Moreover, $\lambda_n \to 0$.