Anderson localization and topological phases

Gian Michele Graf
ETH Zurich

Summer School on ”Operator Algebras, Spectral Theory, and Applications to Topological Insulators”
Tbilisi
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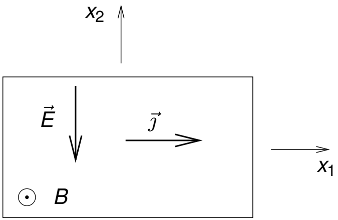
based on joint works with A. Elgart, J. Schenker, M. Porta, J. Shapiro; C. Tauber
and on discussions with Y. Avron, J. Fröhlich

Outline

The phenomenon



B: magnetic field
$\vec{E}$: electric field
$\vec{\jmath}$: current density

Electron gas of
- density n
- Fermi energy μ

Hall-Ohm law

$$\vec{\jmath} = \underline{\sigma}\vec{E}\ , \qquad \underline{\sigma} = \begin{pmatrix} \sigma_{\mathrm{D}} & \sigma_{\mathrm{H}} \\ -\sigma_{\mathrm{H}} & \sigma_{\mathrm{D}} \end{pmatrix}$$

σ_{H}: Hall conductance
σ_{D}: dissipative conductance, ideally $= 0$

The experiment (von Klitzing, 1980)

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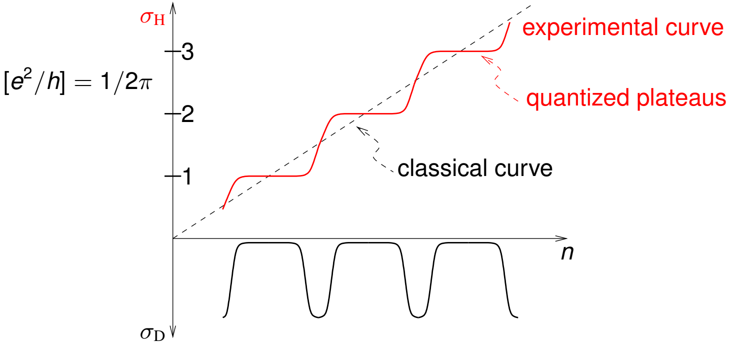

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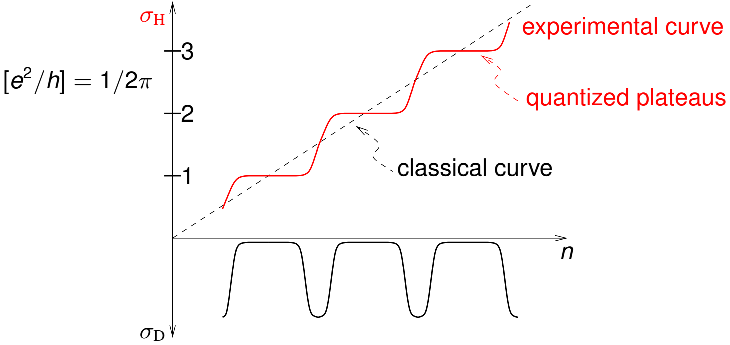

Fractional Quantum Hall effect not discussed

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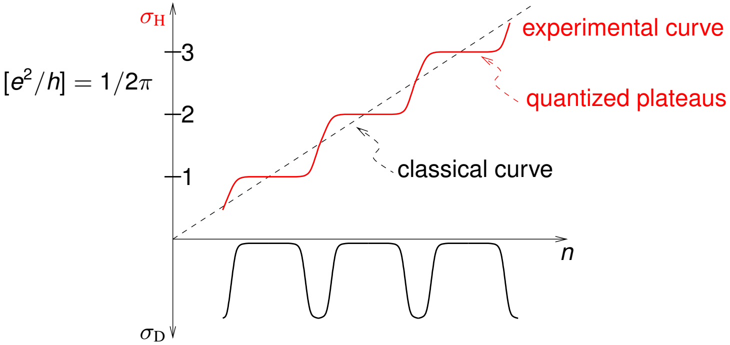

Width of plateaus increases with disorder

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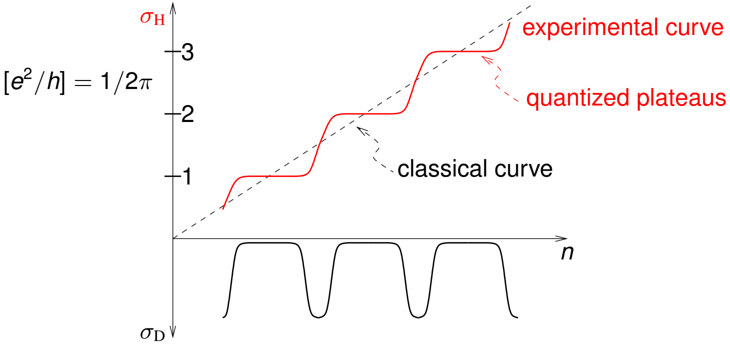
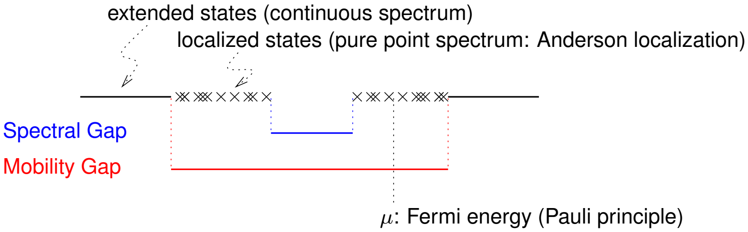


Experiment: $h/e^2 = 25'812.807'4555(59)$ Ohm

Spectral vs. Mobility Gap

The spectrum of a single-particle Hamiltonian



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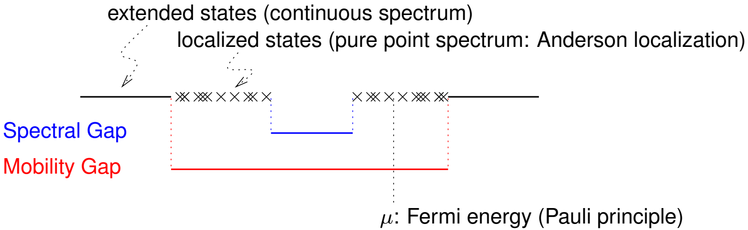

- (integrated) density of states $n(\mu)$ is constant for μ in a Spectral Gap, and strictly increasing otherwise

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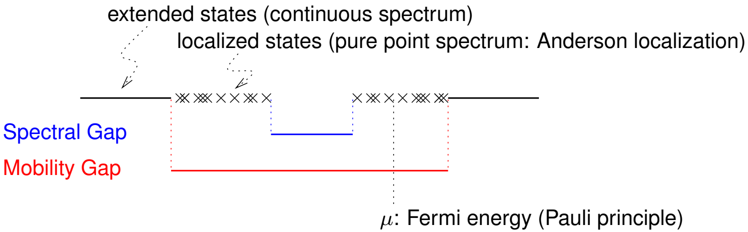

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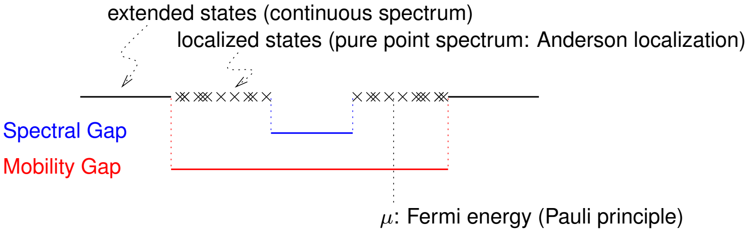

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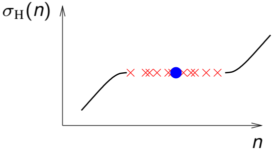
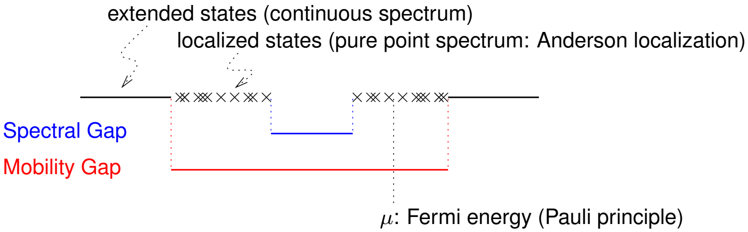


Plateaus arise because of a Mobility Gap only!

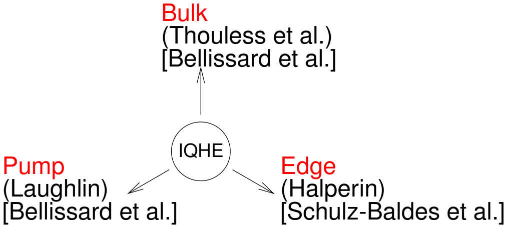
The role of disorder

The spectrum of a single-particle Hamiltonian

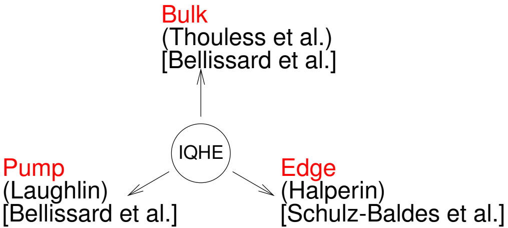


- For a periodic (crystalline) medium:
  - Method of choice: Bloch theory and vector bundles (Thouless et al.)
  - Gap is spectral
- For a disordered medium:
  - Method of choice: Non-commutative geometry (Bellissard; Avron et al.)
  - Fermi energy may lie in a mobility gap (better) or just in a spectral gap

Interpretations of IQHE and definitions of $\sigma_{\rm H}$

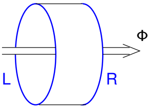


Interpretations of IQHE and definitions of σ_{H}



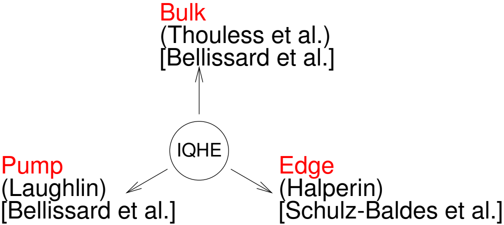
Pump:

$2\pi\sigma_{\mathrm{P}} \equiv$ number *n* of electrons pumped from L to R upon increasing the magnetic flux Φ by 2π. (Note: $\Phi \rightsquigarrow \Phi + 2\pi$ implies $H \rightsquigarrow UHU^*$.)



Quantization: *n* is an integer.

Interpretations of IQHE and definitions of σ_{H}

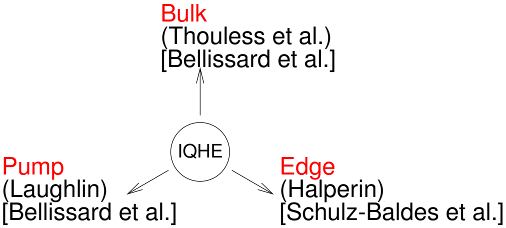


Bulk:

σ_{B} conductivity by Kubo formula: Current density $\vec{\jmath}$ as linear response to an applied (weak) electric field $\vec{E}$ in the bulk.

Quantization: $2\pi\sigma_{\mathrm{B}}$ is a Chern number.

Interpretations of IQHE and definitions of σ_{H}

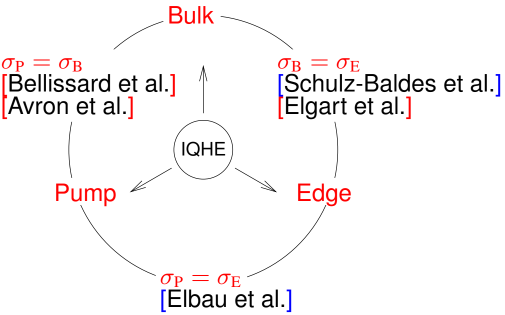


Edge:

σ_{E} conductance: Current carried by edge states per unit voltage, $\sigma_{\mathrm{E}} = dI/d\mu$.

Quantization: $2\pi\sigma_{\mathrm{E}}$ is the number of edge channels.

Equivalences of interpretations

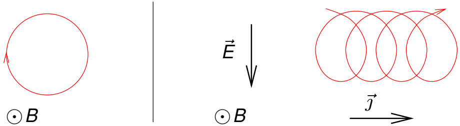


[ ]: spectral gap

[ ]: mobility gap

Bulk vs. Edge

- (Quantum) Hall as a bulk effect



A voltage difference entails an electric field in the bulk

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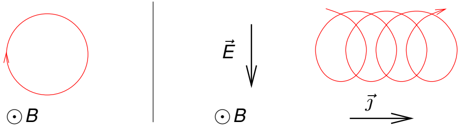

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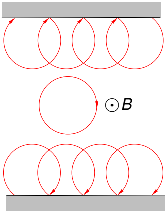

A voltage difference entails different Fermi energies of (chiral) edge states at opposite edges

Heuristic argument for $\sigma_{\mathrm{B}} = \sigma_{\mathrm{E}}$

Bulk: $\vec{\jmath} = -\sigma_{\mathrm{B}} \varepsilon \vec{E}$ with $\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (rotation by $\pi/2$)

Edge: $\sigma_{\mathrm{E}} = dI/d\mu$, i.e. $I = \sigma_{\mathrm{E}}(\mu - \varphi)$ with Fermi energy μ and electric potential φ at the edge

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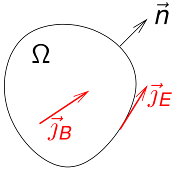

Notation: χ_Ω indicator function of Ω, $\delta_{\partial\Omega}$ delta distribution on $\partial\Omega$, $\vec{n}$ normal vector

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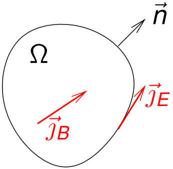

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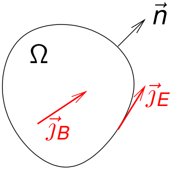

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$$\vec{\jmath}_B = -\chi_\Omega\sigma_{\mathrm{B}}\varepsilon\vec{E} \qquad\qquad \vec{\jmath}_E = \sigma_{\mathrm{E}}(\mu - \varphi)\varepsilon\vec{n}\delta_{\partial\Omega}$$

$$= \chi_\Omega\sigma_{\mathrm{B}}\varepsilon\vec{\nabla}\varphi \qquad\qquad = -\sigma_{\mathrm{E}}(\mu - \varphi)\varepsilon\vec{\nabla}\chi_\Omega$$

$$\mathrm{div}(\varepsilon\vec{v}) = -\,\mathrm{curl}\,\vec{v} \;\; (= 0 \text{ for } \vec{v} = \vec{\nabla}\varphi)$$

$$\mathrm{div}\,\vec{\jmath}_B = \sigma_{\mathrm{B}}\vec{\nabla}\chi_\Omega \cdot \varepsilon\vec{\nabla}\varphi$$

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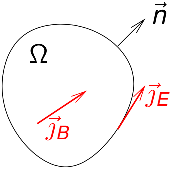

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$$\vec{\jmath}_B = -\chi_\Omega\sigma_{\rm B}\varepsilon\vec{E} = \chi_\Omega\sigma_{\rm B}\varepsilon\vec{\nabla}\varphi \qquad \vec{\jmath}_E = \sigma_{\rm E}(\mu-\varphi)\varepsilon\vec{n}\delta_{\partial\Omega} = -\sigma_{\rm E}(\mu-\varphi)\varepsilon\vec{\nabla}\chi_\Omega$$

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$$\mathrm{div}\,\vec{\jmath}_B = \sigma_{\rm B}\vec{\nabla}\chi_\Omega\cdot\varepsilon\vec{\nabla}\varphi$$

$$\mathrm{div}\,\vec{\jmath}_E = \sigma_{\rm E}\vec{\nabla}\varphi\cdot\varepsilon\vec{\nabla}\chi_\Omega$$

Thus $\mathrm{div}(\vec{\jmath}_B + \vec{\jmath}_E) = 0$ implies $\sigma_{\rm E} = \sigma_{\rm B}$.

Topological insulators: definition stated

- Insulator in the Bulk: Excitation gap
  For independent electrons: spectral gap at Fermi energy μ

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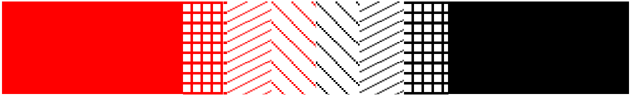
- Integer QHE: $2\pi\sigma_{\mathrm{H}} \in \mathbb{Z}$ tells classes apart
- Refinement: The Hamiltonians enjoy a symmetry which is preserved under deformations. (Classification trivially more restrictive, yet potentially richer: Hamiltonians along deformation may not enjoy symmetry even if endpoints do. Thus finer classes.)

Bulk-edge correspondence

Recall: In the space of Hamiltonians, a topological insulator can not be deformed in an ordinary one, while keeping the gap open and respecting symmetries

Bulk-edge correspondence

Deformation as interpolation in physical space:



- Gap must close somewhere in between. Hence: Interface states at Fermi energy.

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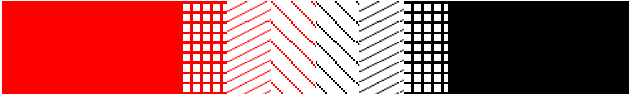

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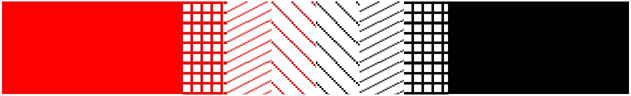

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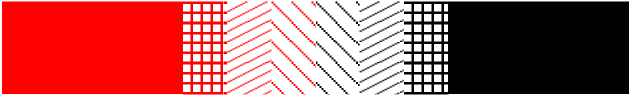

- Gap must close somewhere in between. Hence: Interface states at Fermi energy.
- Ordinary insulator $\rightsquigarrow$ void: Edge states
- Bulk-edge correspondence: Termination of bulk of a topological insulator implies edge states. (But not conversely!)

Bulk-edge correspondence

In a nutshell: Termination of bulk of a topological insulator implies edge states

- Topological insulators are insulating in the bulk, but conducting on the surface

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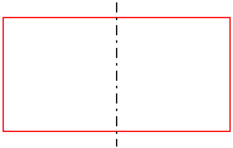

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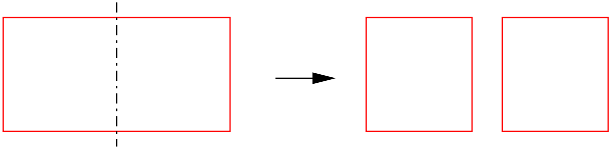

The periodic table of topological matter

Symmetry Class	Θ	Σ	Π	d = 1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Notation for symmetries:

- Θ (time-reversal): antiunitary, $H\Theta = \Theta H$, $\Theta^2 = \pm 1$
- Σ (charge-conjugation): antiunitary, $H\Sigma = -\Sigma H$, $\Sigma^2 = \pm 1$
- $\Pi = \Theta\Sigma = \Sigma\Theta$: unitary

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D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

First version: Schnyder et al.; then Kitaev based on Altland-Zirnbauer; based on Bloch theory

The periodic table of topological matter

Symmetry				d							
Class	Θ	Σ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

By now: Non-commutative (bulk) index formulae have been found in all cases (Prodan, Schulz-Baldes)

Special cases to be considered

Symmetry Class	Θ	Σ	Π	d = 1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

...and one more

Various approaches to the QHE

- Landau Hamiltonians (not discussed)
- Periodic Hamiltonians (Thouless et al.)
- The role of disorder and non-commutative geometry
- Effective field theories (important, but not discussed; Fröhlich et al.)

Broad mathematical setting

Definitions of σ_{H} and their equivalences should

• be based on a microscopic model (Schrödinger operator), as opposed to an effective theory (conformal or topological field theory).

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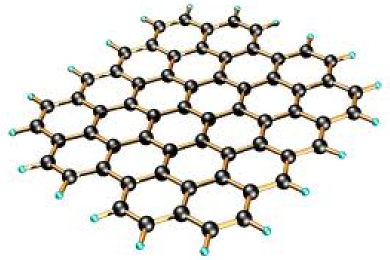
Plane: lattice $\Gamma \ni x = (x_1, x_2)$, e.g. $\Gamma = \mathbb{Z}^2$

Single-particle Hamiltonian H_B: operator on $\ell^2(\Gamma)$ with $H_B(x', x)$ of short range in $|x - x'|$ (tight binding model).



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• apply to infinite systems (thermodynamic limit)

• preferably, be compatible with disorder: Fermi energy μ lies in a Mobility Gap (as opposed to a Spectral Gap).

Mobility gap, technically speaking

Hamiltonian H_B on $\ell^2(\mathbb{Z}^d)$
$P_\mu = E_{(-\infty,\mu)}(H_B)$ Fermi projection,

Assumption. Fermi projection has strong off-diagonal decay:

$$\sup_{x'} \mathrm{e}^{-\varepsilon|x'|} \sum_x \mathrm{e}^{\nu|x-x'|} |P_\mu(x,x')| < \infty$$

(some $\nu > 0$, all $\varepsilon > 0$)

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- Trivially true for H_B a multiplication operator in position space
- Trivially false for H_B a function of momentum ($P_\mu(x,0) \sim |x|^{-d}$)
- Proven in (virtually) all cases where localization is known.

Mobility gap and dynamical localization (DL)

DL of a random Schrödinger operator H_ω, $(\omega \in \Omega)$ in an interval Δ means (or could equivalently mean) that for some $\nu > 0$ (Notation: $K(x, x') = \langle x|K|x'\rangle$)

$$\mathbb{E}\Big(\sup_{g \in B_1(\Delta)} |\langle x|g(H_\omega)|x'\rangle| \Big) \leq C\mathrm{e}^{-2\nu|x-x'|}$$

where

$$B_1(\Delta) = \{g : \mathbb{R} \to \mathbb{C} \mid |g(\lambda)| \leq 1, g \text{ constant on } \lambda \gtrless \Delta\}$$

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Let $g(\lambda) = \mathrm{e}^{-\mathrm{i}t\lambda}E_\Delta(\lambda)$ ($\in B_1(\Delta)$) for $t \in \mathbb{R}$. By DL

$$\mathbb{E}\big(\sup_{t\in\mathbb{R}} |\langle x|\mathrm{e}^{-\mathrm{i}tH_\omega}E_\Delta(H_\omega)|x'\rangle|\big) \le C\mathrm{e}^{-2\nu|x-x'|}$$

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- explains name ”DL”
- implies spectral localization

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$$\mathbb{E}\Big(\sum_{x,x' \in \mathbb{Z}^d} |\langle x|P_{\mu,\omega}|x'\rangle| \mathrm{e}^{\nu|x-x'|} \mathrm{e}^{-\varepsilon|x'|} \Big) \le C < +\infty$$

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In particular (drop $\mathbb{E}$, $\sum_{x'}$)

$$\mathrm{e}^{-\varepsilon|x'|} \sum_{x} |\langle x|P_{\mu,\omega}|x'\rangle| \mathrm{e}^{\nu|x-x'|} \le C_\omega < +\infty$$

Aside: Rate of change in QM

State space $\mathcal{H}$ state ψ, observable $X = X^*$. Expectation value is

$$(\psi, X\psi)$$

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Because evolution is $\psi \mapsto \mathrm{e}^{-\mathrm{i}Ht}\psi$, so

$$\frac{d}{dt}(\mathrm{e}^{-\mathrm{i}Ht}\psi, X\mathrm{e}^{-\mathrm{i}Ht}\psi)\big|_{t=0} = (\psi, \mathrm{i}[H, X]\psi)$$

Aside: Poor man's second quantization for fermions

Single particle Hilbert space $\mathcal{H} \in \psi$

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Many particle state *S* has single-particle marginal ("density matrix") ρ: operator on $\mathcal{H}$

$$\rho = \rho^* , \qquad 0 \leq \rho \leq 1$$

Meaning: ρ tells expected occupation of any single-particle state $\psi \in \mathcal{H}$, ($(\psi, \psi) = 1$) in the state *S* as

$$(\psi, \rho\psi) = \mathrm{tr}(P\rho) \qquad (\in [0, 1])$$

with $P = \psi(\psi, \cdot)$ the projection onto ψ.

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Expectation value in S:

$$\sum_i x_i \,\mathrm{tr}(P_i \rho) = \mathrm{tr}(X\rho)$$

Aside: Gauge transformations

(Units $e = \hbar = c = 1$)

Electromagnetic(e.m.) fields $\vec{E} = \vec{E}(\vec{x}, t)$, $\vec{B} = \vec{B}(\vec{x}, t)$ expressed in terms of e.m. potentials $\varphi = \varphi(\vec{x}, t)$, $\vec{A} = \vec{A}(\vec{x}, t)$

$$\vec{E} = -\vec{\nabla}\varphi - \partial\vec{A}/\partial t\,, \quad \vec{B} = \text{curl}\,\vec{A}$$

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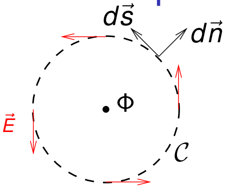
Time-independent gauge transformations are realized as unitaries $U : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \psi \mapsto \mathrm{e}^{\mathrm{i}\chi}\psi$

$$H \mapsto UHU^* = \mathrm{e}^{\mathrm{i}\chi}H\mathrm{e}^{-\mathrm{i}\chi} = H'$$

(by $\mathrm{e}^{\mathrm{i}\chi}(\vec{p} - \vec{A})\mathrm{e}^{-\mathrm{i}\chi} = \vec{p} - \vec{A}'$)

IQHE as a pump: Flux insertion



Flux increase from 0 to Φ
Charge Q traversing $\mathcal{C}$ inwards

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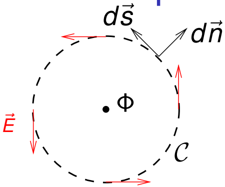

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$$\frac{dQ}{dt} = -\oint_{\mathcal{C}} \vec{\jmath} \cdot d\vec{n} = -\sigma_{\mathrm{H}} \oint_{\mathcal{C}} \vec{E} \cdot d\vec{s} = \sigma_{\mathrm{H}} \frac{d\Phi}{dt}$$

$$Q = \sigma_{\mathrm{H}} \Phi$$

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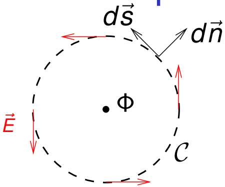

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Flux Φ generated by a gauge potential $\vec{A}$:

$$\oint_{\mathcal{C}} \vec{A}\cdot d\vec{s} = \Phi, \text{ e.g. } \vec{A} = \vec{\nabla}\Big(\frac{\Phi}{2\pi}\arg\vec{x}\Big) \equiv \vec{\nabla}\chi$$

If $\chi(\vec{x})$ were single-valued:

gauge	$\vec{A} = 0$	equiv. to	$\vec{A} = \vec{\nabla}\chi$
	$\downarrow$		$\downarrow$
Hamiltonian	H_B		UH_BU^*

with $U = \mathrm{e}^{\mathrm{i}\chi}$, unitary. For $\Phi = 2\pi$, U is single-valued, though $\chi(\vec{x}) = \arg\vec{x}$ is not.

Charge Q according to quantum mechanics

Fermi energy μ: all single-particle eigenstates of H_B with eigenvalues (energies) $\leq \mu$ are occupied

Fermi projection (FP) of H_B ($\Phi = 0$): $P_\mu = E_{(-\infty,\mu)}(H_B)$

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Evolution of FP as flux $\Phi(t)$ increases from 0 to 2π: $\tilde{U}P_\mu\tilde{U}^*$ with propagator $\tilde{U}$

Tentatively, the charge Q is

$$2\pi\sigma_{\mathrm{P}} = \text{“} \dim \tilde{U}P_\mu\tilde{U}^* - \dim UP_\mu U^* \text{”} = \infty - \infty$$

($\dim P = \dim \operatorname{Ran} P$). The (non existent) expression counts difference in number of electrons: After pumping to $\Phi = 2\pi$, resp. in equilibrium at $\Phi = 2\pi$.

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Rightly interpreted, it is an integer. Hence

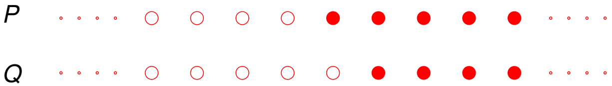
$$2\pi\sigma_{\mathrm{P}} = \text{“}\ \dim P_\mu - \dim UP_\mu U^*\ \text{”}$$

since $\tilde{U}$ is connected to 1 (unlike U)

The index of a pair of projections

Orthogonal projections P, Q on a Hilbert space $\mathcal{H}$.

Example (Hilbert's hotel): $\mathcal{H} = \ell^2(\mathbb{Z})$, projections P, Q defined by filled dots $n \in \mathbb{Z}$.



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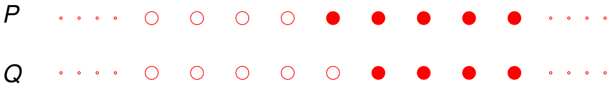

Generalizations of $\dim P - \dim Q$:

$$\operatorname{tr}(P - Q)$$

since $\operatorname{tr} P = \dim P$

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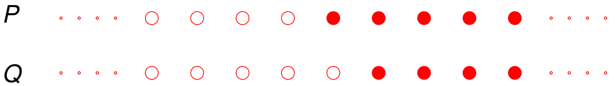

Generalizations of $\dim P - \dim Q$:

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since $\operatorname{tr} P = \dim P$. More generally:
Definition. The Index of a pair of projections is

$$\begin{aligned}\operatorname{Ind}(P, Q) = \dim\{\psi \in \mathcal{H} \mid P\psi = \psi, Q\psi = 0\} + \\ - \dim\{\psi \in \mathcal{H} \mid Q\psi = \psi, P\psi = 0\}\end{aligned}$$

(if dimensions finite)
Remarks. (i) In the example, both generalizations $= 1$. (ii) In the IQHE only the index is well-defined

Properties of the Index

- Additivity: $\mathrm{Ind}(P, Q) = \mathrm{Ind}(P, R) + \mathrm{Ind}(R, Q)$
- Stability: $\|P - Q\| < 1 \;\Rightarrow\; \mathrm{Ind}(P, Q) = 0$
- $$\mathrm{Ind}(P, Q) = \mathrm{tr}(P - Q)^{2n+1}$$

  if $P - Q \in \mathcal{J}_{2n+1}$ (trace ideals).

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Remarks. (i) $\mathrm{Ind}(P, Q) = \dim P - \dim Q$ (finite-dimensional case)
(ii) $\mathrm{tr}(P - Q)^3 = \mathrm{tr}(P - Q)$ if $P - Q \in \mathcal{J}_1$; because

$$(P - Q) - (P - Q)^3 = [PQ, [Q, P - Q]]$$

$$AB, BA \in \mathcal{J}_1 \;\Rightarrow\; \mathrm{tr}[A, B] = 0$$

(iii) $\mathrm{Ind}(P, Q) = \mathrm{ind}(QP)$ as a map on $\mathrm{ran}\, P \to \mathrm{ran}\, Q$

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(iv) If the unitary U has an eigenbasis and $P - UPU^* \in \mathcal{J}_1$, then $\mathrm{tr}(P - UPU^*) = 0$. In fact, by $U\psi_n = u_n\psi_n$

$$(\psi_n, (P - UPU^*)\psi_n) = (1 - |u_n|^2)(\psi_n, P\psi_n) = 0$$

IQHE as a pump: Definition of σ_{P}

Definition.

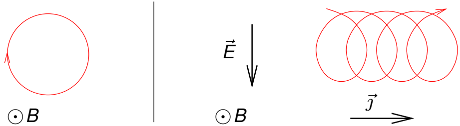
$$2\pi\sigma_{\mathrm{P}} = \mathrm{Ind}(P_\mu, UP_\mu U^*) \qquad \text{(Bellissard)}$$
$$= \mathrm{tr}(P_\mu - UP_\mu U^*)^3 \qquad \text{(Avron et al.)}$$

where $U = \arg\vec{x} = z/|z|$.

Remarks. (i) Is a (stable) integer, whenever defined.
(ii) $P_\mu - UP_\mu U^* \notin \mathcal{J}_1$.

IQHE as a Bulk effect

Example: Cyclotron orbit drifting under a electric field $\vec{E}$

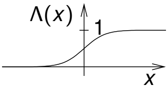


General: Hamiltonian H_B in the plane. Kubo formula (linear response to $\vec{E}$)

$$\sigma_{\mathrm{B}} = \mathrm{i}\,\mathrm{tr}\, P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]\big]$$

where

$P_\mu = E_{(-\infty,\mu)}(H_B)$ Fermi projection,
$\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$ switches



IQHE as a Bulk effect (remarks)

Kubo formula (Bellissard et al., Avron et al.)

$$\sigma_{\mathrm{B}} = \mathrm{i}\,\mathrm{tr}\, P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]\big]$$

extends the formula for the periodic case (Thouless et al., Avron)

$$\sigma_{\mathrm{B}} = -\frac{\mathrm{i}}{(2\pi)^2} \int_{\mathbb{T}} d^2k\, \mathrm{tr}(P(k)[\partial_1 P(k), \partial_2 P(k)])$$

where $\mathbb{T}$: Brillouin zone (torus); $P(k)$ Fermi projection on the space of states of quasi-momentum $k = (k_1, k_2)$; $\partial_i = \partial/\partial k_i$

Remarks.

$$2\pi\sigma_{\mathrm{B}} = \mathrm{ch}(E)$$

the Chern number of the vector bundle E over $\mathbb{T}$ and fiber range $P(k)$ (see later)

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Remarks.

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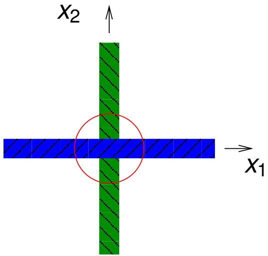
Alternative treatment of disorder (Thouless): Large, but finite system (square); $(k_1, k_2) \rightsquigarrow (\varphi_1, \varphi_2)$ phase slips in boundary conditions

IQHE as a Bulk effect (remarks)

$$\sigma_{\mathrm{B}} = \mathrm{i}\,\mathrm{tr} P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]\big]$$

where $\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$ switches. Supports of $\vec{\nabla}\Lambda_i$:



Recall Kubo: $j_1 = -\sigma_{\mathrm{B}} E_2$

Remarks. (i) Λ_1, Λ_2: where from? Current operator across $x_1 = 0$: $\mathrm{i}[H_B, \Lambda_1]$; field $\vec{E} = -\vec{\nabla}\Lambda_2$

(ii) The trace is well-defined. Roughly: An operator has a well-defined trace if it acts non-trivially on finitely many states only. Here the intersection contains only finitely many sites.

Theorem: Quantization and equivalence

Definition. Ergodic operators H_ω, ($\omega \in \Omega$: probability space): actions of (magnetic) $\mathbb{Z}^2$-translations on Ω and on $\ell^2(\mathbb{Z}^2)$ compatible.

Theorem [Index=2π Kubo] (Bellissard, van Elst, Schulz-Baldes)
If μ lies in a Mobility Gap, then $\sigma_{\mathrm{D}}(\mu) = 0$ and $2\pi\sigma_{\mathrm{P}}(\mu) = 2\pi\sigma_{\mathrm{B}}(\mu)$ is an integer and constant.

Proof by non-commutative geometry.

Theorem and proof reformulated

Theorem [Index=2π Kubo] (Avron, Seiler, Simon)
If μ lies in a Mobility Gap, then $2\pi\sigma_{\mathrm{P}} = 2\pi\sigma_{\mathrm{B}}$, i.e.

$$\mathrm{tr}(P_\mu - UP_\mu U^*)^3 = 2\pi\mathrm{i}\ \mathrm{tr}\, P_\mu[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]$$

Remark. No ergodic setting.

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Explicitely,

$$2\mathrm{i} \sum_{x,y,z\in\mathbb{Z}^2} P_\mu(x,y)P_\mu(y,z)P_\mu(z,x)S(x,y,z) =$$

$$-2\pi\mathrm{i} \sum_{x,y,z\in\mathbb{Z}^2} P_\mu(x,y)P_\mu(y,z)P_\mu(z,x)[(\Lambda_1(y)-\Lambda_1(x))(\Lambda_2(z)-\Lambda_2(y)) - (1\leftrightarrow 2)]$$

where

$$S(x,y,z) = -\frac{\mathrm{i}}{2}\left(1-\frac{U(x)}{U(y)}\right)\left(1-\frac{U(y)}{U(z)}\right)\left(1-\frac{U(z)}{U(x)}\right)$$
$$= \sin\angle(x,0,y) + \sin\angle(y,0,z) + \sin\angle(z,0,x)$$

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$$= \sin\angle(x,0,y) + \sin\angle(y,0,z) + \sin\angle(z,0,x)$$

Remark. Mobility gap: Substantial contribution only when x, y, z all near 0.

Sketch of proof

• Flux and cross are centered at the origin $p = 0$. Take instead $p \in \mathbb{R}^2$ arbitrary: neither side changes. For $w = x, y, z$ replace

$$\Lambda_i(w) \rightsquigarrow \Lambda_i(w - p), \qquad U(w) \rightsquigarrow U(w - p)$$

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- Average both sides over $p \in C_L$ (cube of side L):

$$L^{-2} \int_{p \in C_L} d^2 p$$

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(by mobility gap) for L large

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- $(p, y, x \in \mathbb{R})$

$$\int dp (\Lambda(y - p) - \Lambda(x - p)) = y - x$$

because $= f(y - x)$, $f(0) = 0$ and $f'(y - x) = \int dp \, \Lambda'(y - p) = 1$.

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- On r.h.s. use

$$\int dp_1 dp_2 (\Lambda(y_1 - p_1) - \Lambda(x_1 - p_1))(\Lambda(z_2 - p_2) - \Lambda(y_2 - p_2)) - (1 \leftrightarrow 2)$$

$$= (y_1 - x_1)(z_2 - y_2) - (1 \leftrightarrow 2) = 2\,\mathrm{Area}(x, y, z)$$

Sketch of proof (continued)

The claim

$$2\mathrm{i}\sum_{x,y,z\in\mathbb{Z}^2} P_\mu(x,y)P_\mu(y,z)P_\mu(z,x)S(x,y,z) =$$

$$-2\pi\mathrm{i}\sum_{x,y,z\in\mathbb{Z}^2} P_\mu(x,y)P_\mu(y,z)P_\mu(z,x)[(\Lambda_1(y)-\Lambda_1(x))(\Lambda_2(z)-\Lambda_2(y))-(1\leftrightarrow 2)]$$

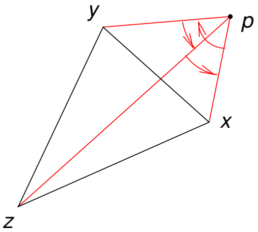
reduces by the above to

$$\int d^2p(\sin\angle(x,p,y)+\sin\angle(y,p,z)+\sin\angle(z,p,x)) = 2\pi\,\mathrm{Area}(x,y,z)$$

Sketch of proof (continued)

$$\int d^2p(\sin\angle(x,p,y)+\sin\angle(y,p,z)+\sin\angle(z,p,x)) = 2\pi\,\mathrm{Area}(x,y,z)$$

(Connes' triangle formula)



Sketch of proof (continued)

$$\int d^2p(\sin\angle(x,p,y) + \sin\angle(y,p,z) + \sin\angle(z,p,x)) = 2\pi\,\mathrm{Area}(x,y,z)$$

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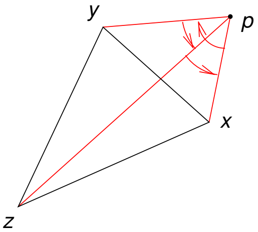

Proof: Observation (Colin de Verdière)

- Drop sin: obvious.

Sketch of proof (continued)

$$\int d^2p(\sin\angle(x,p,y)+\sin\angle(y,p,z)+\sin\angle(z,p,x)) = 2\pi\,\mathrm{Area}(x,y,z)$$

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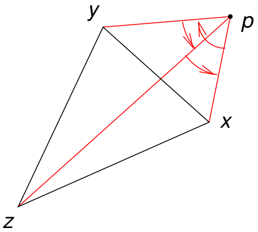


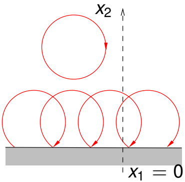
Proof: Observation (Colin de Verdière)

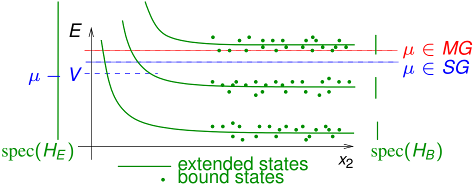
- Drop sin: obvious.
- Let f be odd with $f(t)-t=O(t^3)$, $(t\to 0)$; e.g. $f=\sin$. Then

$$\int d^2p(f(\angle(x,p,y))-\angle(x,p,y)) = 0$$

by (i) integrand $0(|p|^{-3})$, $(p\to\infty)$ and (ii) reflection symmetry.

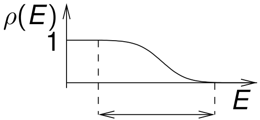
IQHE as an edge effect





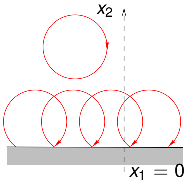
Hamiltonian H_E on the upper half-plane: restriction of H_B through boundary conditions at $x_2 = 0$.

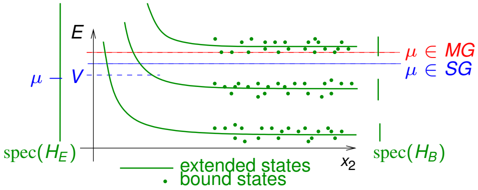
State $\rho(H_E)$: 1-particle density matrix, e.g. $\rho(H_E) = E_{(-\infty,\mu)}(H_E)$, or (actually) smooth



supp $\rho' \subset$ Spectral Gap for H_B (not for H_E)

IQHE as an edge effect





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State $\rho(H_E)$: 1-particle density matrix, e.g. $\rho(H_E) = E_{(-\infty,\mu)}(H_E)$, or (actually) smooth

Current operator across $x_1 = 0$: $\mathrm{i}[H_E, \Lambda_1]$

$$I = \mathrm{i}\ \mathrm{tr}(\rho(H_E + V) - \rho(H_E))[H_E, \Lambda_1]$$

As $V \to 0$: $I/V \to \sigma_{\mathrm{E}}$

$$\sigma_{\mathrm{E}} = \mathrm{i}\ \mathrm{tr}(\rho'(H_E)[H_E, \Lambda_1])$$

Equality of conductances

Theorem (Schulz-Baldes, Kellendonk, Richter). Ergodic setting. If the Fermi energy μ lies in a Spectral Gap of H_B, then

$$\sigma_{\mathrm{E}} = \sigma_{\mathrm{B}}.$$

In particular, σ_{E} does not depend on ρ', nor on boundary conditions.

What about the case of a Mobility Gap?

Is

$$\sigma_{\mathrm{E}} = -\mathrm{i}\,\mathrm{tr}(\rho'(H_E)[H_E, \Lambda_1])$$

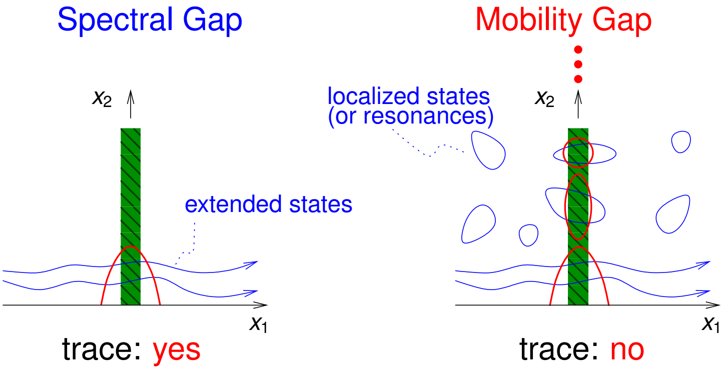
well-defined? (Here, switches Λ_i ($i = 1, 2$) with flipped orientations)

What about the case of a Mobility Gap?

Is

$$\sigma_{\mathrm{E}} = -\mathrm{i}\,\mathrm{tr}(\rho'(H_E)[H_E, \Lambda_1])$$

well-defined?



$\therefore$ the definition of σ_{E} needs to be changed in case of a Mobility Gap!

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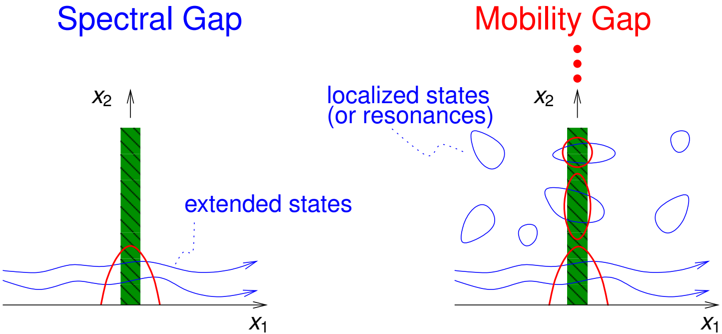


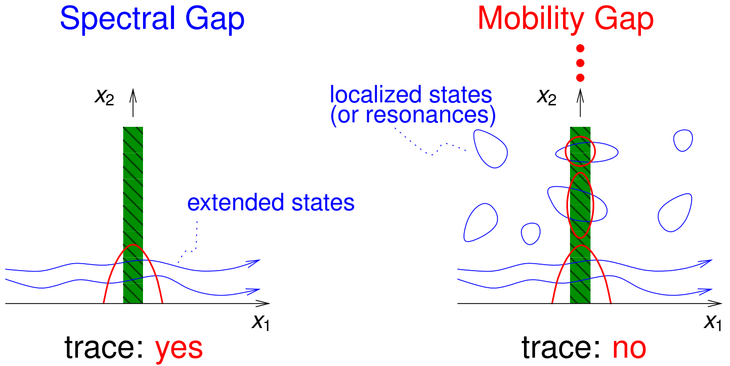
trace: yes

trace: no

$\therefore$ the definition of σ_{E} needs to be changed in case of a Mobility Gap!

Guiding principle: Localized states should not contribute to the edge current

What about the case of a Mobility Gap?



$\therefore$ the definition of σ_{E} needs to be changed in case of a Mobility Gap!

Analogy: Electrodynamics of continuous media

$$\vec{\jmath} = \vec{\jmath}_F + \vec{\jmath}_M \equiv \text{free} + \text{molecular currents} \qquad \vec{\jmath}_M = \text{curl}\, \vec{M}$$

Localized states should not contribute to the (free) edge current

Equality of conductances

For a so amended definition of σ_{E}:

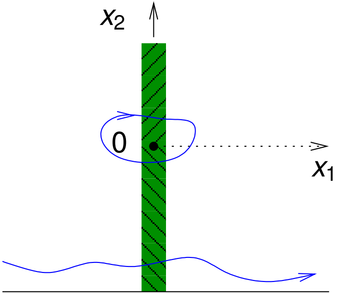
Theorem (Elgart, G., Schenker). If $\operatorname{supp} \rho'$ lies in a Mobility Gap, then

$$\sigma_{\mathrm{E}} = \sigma_{\mathrm{B}}$$

In particular σ_{E} does not depend on ρ', nor on boundary conditions.

Definition of σ_{E} in case of a Mobility Gap

Replace H_E to H_a ($a > 0$) as follows



edge: $x_2 = -a$
(eventually: $-a \to -\infty$)

- Current across the portion ▨ of $x_1 = 0$:

$$-\mathrm{i}\,\mathrm{tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2) \quad \text{(exists!)}$$

- Current across the portion ▨:

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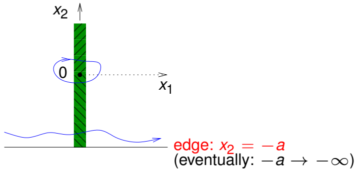

- Current across the portion of $x_1 = 0$:

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- Current across the portion: In the limit $a \to \infty$ pretend that

$$\rho'(H_a) \rightsquigarrow \rho'(H_B) = \sum_\lambda \rho'(\lambda)\psi_\lambda(\psi_\lambda, \cdot)$$

(sum over eigenvalues λ of H_B: $H_B\psi_\lambda = \lambda\psi_\lambda$)

$$(\psi_\lambda, [H_B, \Lambda_1](1 - \Lambda_2)\psi_\lambda) = -(\psi_\lambda, [H_B, \Lambda_1]\Lambda_2\psi_\lambda)$$

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$$(\psi_\lambda, [H_B, \Lambda_1](1 - \Lambda_2)\psi_\lambda) = -(\psi_\lambda, [H_B, \Lambda_1]\Lambda_2\psi_\lambda)$$

- Together:

$$\sigma_{\mathrm{E}} = \lim_{a\to\infty} -\mathrm{i}\,\mathrm{tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2) + \\ + \mathrm{i}\sum_{\lambda} \rho'(\lambda)(\psi_\lambda, [H_B, \Lambda_1]\Lambda_2\psi_\lambda)$$

Magnetization

Question? What is the term

$$\mathrm{i}(\psi_\lambda, [H_B, \Lambda_1]\Lambda_2\psi_\lambda) \ ?$$

Magnetization

Question? What is the term

$$\mathrm{i}(\psi_\lambda, [H_B, \Lambda_1]\Lambda_2\psi_\lambda) \; ?$$

Or better after hermitization of $\mathrm{i}[H_B, \Lambda_1]\Lambda_2$, i.e.

$$\frac{\mathrm{i}}{2}([H_B, \Lambda_1]\Lambda_2 - \Lambda_2[\Lambda_1, H_B]) = \frac{\mathrm{i}}{2}[H_B, \Lambda_1\Lambda_2] - \frac{\mathrm{i}}{2}(\Lambda_1 H_B \Lambda_2 - \Lambda_2 H_B \Lambda_1)$$

where we get

$$-\frac{\mathrm{i}}{2}(\psi_\lambda, (\Lambda_1 H_B \Lambda_2 - \Lambda_2 H_B \Lambda_1)\psi_\lambda) \; ?$$

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Answer: Replacement $x_i \rightsquigarrow \Lambda_i$, $(i = 1, 2)$ signifies extensive $\rightsquigarrow$ intensive. Thus

$$m = \frac{1}{2}\vec{x} \wedge \dot{\vec{x}} \rightsquigarrow M = \frac{1}{2}(\Lambda_1 \dot{\Lambda}_2 - \Lambda_2 \dot{\Lambda}_1)$$

signifies "magnetic moment $\rightsquigarrow$ magnetization". So, by $\dot{\Lambda}_i = \mathrm{i}[H_B, \Lambda_i]$,

$$M = \frac{\mathrm{i}}{2}(\Lambda_1 H_B \Lambda_2 - \Lambda_2 H_B \Lambda_1)$$

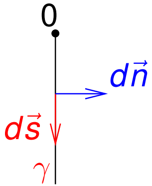
$\therefore$

$$-\frac{\mathrm{i}}{2}(\psi_\lambda, (\Lambda_1 H_B \Lambda_2 - \Lambda_2 H_B \Lambda_1)\psi_\lambda) = -(\psi_\lambda, M\psi_\lambda)$$

Magnetization (alternate)

Magnetization current: $\vec{\jmath}_M = \text{curl}\, M = -\varepsilon \vec{\nabla} M$

- Classically: Magnetization is current across Dirac string γ ($d\vec{n} = \varepsilon d\vec{s}$)



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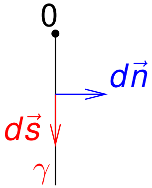

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- Quantum:

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Then hermitize

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- Position space $X = \mathbb{R}^2$ or $X = \mathbb{Z}^2$

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Note: A state $\mathbb{T} \ni k \mapsto \psi_k \in \mathfrak{h}$ is a section of the (trivial) vector bundle $\mathbb{T} \times \mathfrak{h}$

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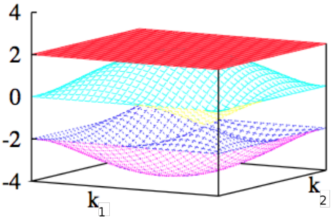

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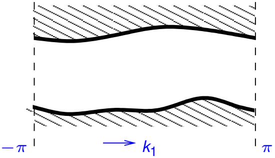

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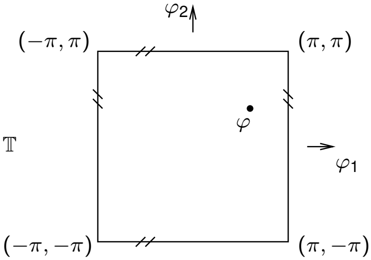
Note: It is a subbundle of $\mathbb{T} \times \mathfrak{h}$, possibly not trivial.

Bundles $(E, \mathbb{T})$ on the 2-torus



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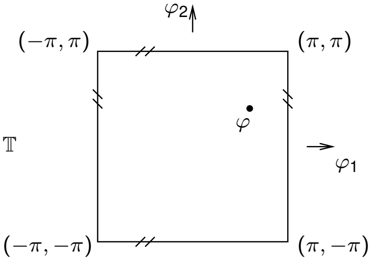

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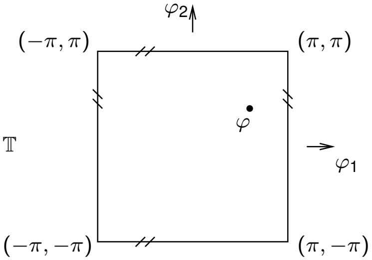

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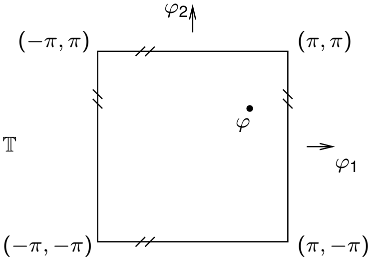

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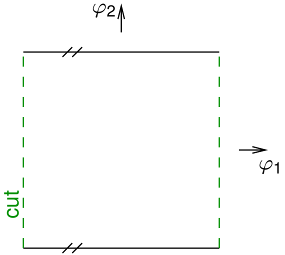
Classification by a Chern number



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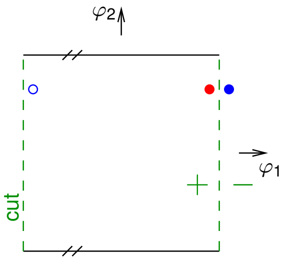

Lemma. On the cut torus the frame bundle admits a section $\varphi \mapsto v(\varphi) \in F(E)_\varphi$

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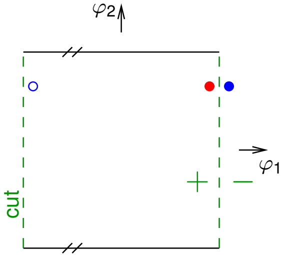

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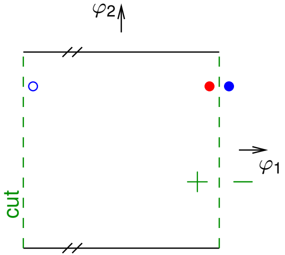

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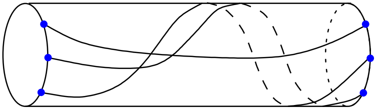

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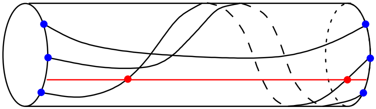

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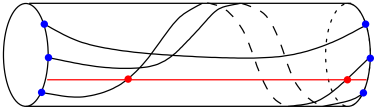

winding number=
signed number of crossings of fiducial line
$\mathrm{ch}(E) = -2$

Hall conductance (bulk)

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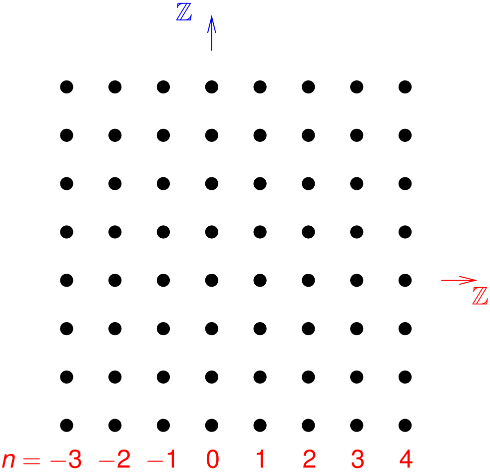
Remark.

$$\mathrm{ch}(E) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} d^2k \, \mathrm{tr}(P(k)[\partial_1 P(k), \partial_2 P(k)])$$

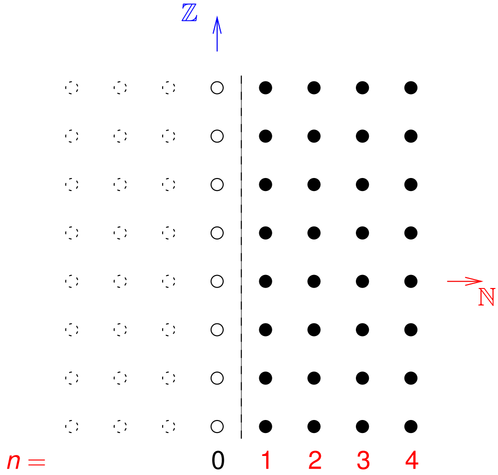
From plane (bulk) to half-plane (edge)

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)



From plane (bulk) to half-plane (edge)

Hamiltonian on the lattice $\mathbb{N} \times \mathbb{Z}$ (half-plane) with $\mathbb{N} = \{1, 2, \ldots\}$



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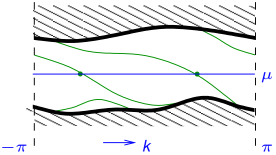
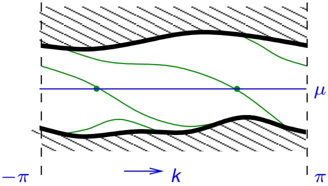
- $H^\sharp(k)$ acting on $L^2(\mathcal{C}^\sharp)$ has continuous and (possibly) discrete spectrum



Hall conductance (edge)



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$\mathcal{N}^\sharp$ = signed number of eigenvalue crossings of Fermi energy

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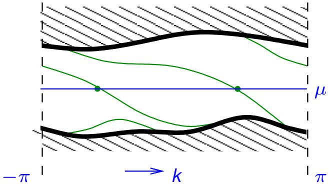

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Physical meaning: The Hall conductance in the edge interpretation is

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(cf. Hatsugai)

Topological insulators: time-reversal invariant case

- Insulator in the Bulk: Excitation gap
  For independent electrons: spectral gap at Fermi energy
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- Time-reversal invariant fermionic system
- Topology: In the space of Hamiltonians, a topological insulator can not be deformed in an ordinary one, while keeping the gap open and time-reversal invariance.

Time-reversal invariance explained

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Remark: By $\Theta E = E$ and $\mathrm{ch}(\Theta E) = -\mathrm{ch}(E)$:

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Such insulators are trivial from the Quantum Hall point of view. Yet interesting in their own class.

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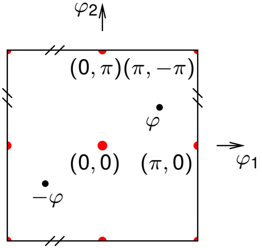
- Kane, Mele; Fu, Kane: Index realized as Pfaffian

Time reversal invariant bundles $(E, \mathbb{T}, \Theta)$

$\mathbb{T}$



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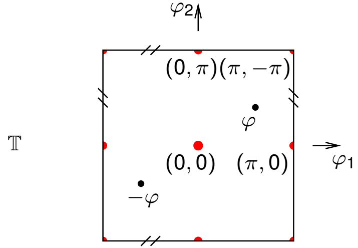

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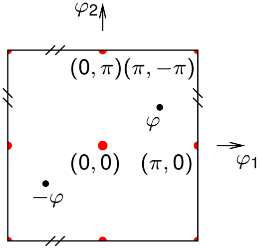

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$$W_{ij}(\varphi) := \langle u_i(\varphi), \Theta u_j(-\varphi) \rangle$$

- Note $W(\varphi)^* W(\varphi) = 1$ and $W(\varphi)^T = -W(-\varphi)$. In particular $W(\varphi)$ antisymmetric at TRIPs.

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- Set

$$\mathcal{I}(E) := \prod_{a \in TRIP} \frac{\text{pf } W(\varphi_a)}{\sqrt{\det W(\varphi_a)}} = \pm 1$$

(Pfaffian defined for antisymmetric matrices, $\det W = (\text{pf } W)^2$)

The Fu-Kane index restated

- Family of matrices $W(\varphi_2)$ with single parameter $0 \leq \varphi_2 \leq \pi$, $\det W(\varphi_2) \neq 0$, antisymmetric at endpoints $\varphi_2 = 0, \pi$

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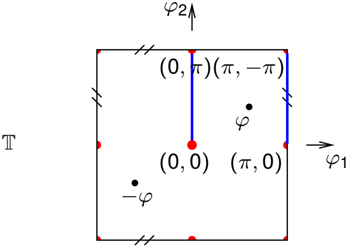

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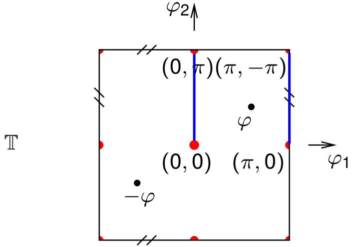

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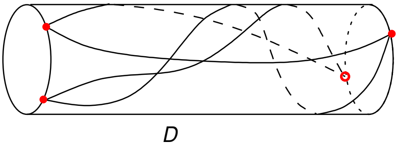
Then

$$\widehat{\mathcal{I}}(E) = \widehat{\mathcal{I}}(W_0)\widehat{\mathcal{I}}(W_\pi)$$

The index of a rueda

Consider a fixed even number of lines moving forward along a (finite) cylinder.

Condition: Lines pair up at the ends



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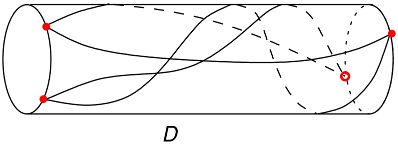

$D = (D(t))_{a \leq t \leq b}$ with $D(t)$ a collection of points on the circle.

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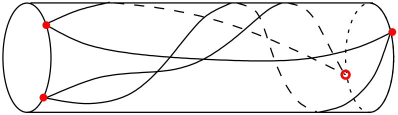

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(Lines can be thought of as world lines of dancers of a rueda)

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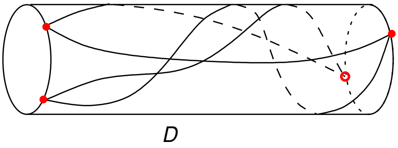

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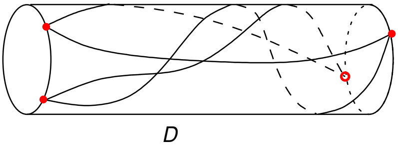

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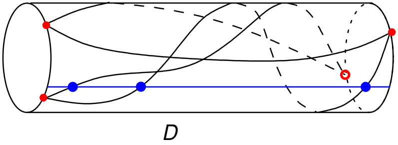

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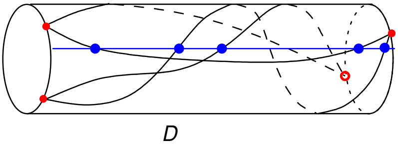

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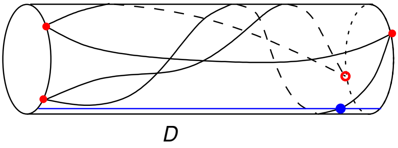

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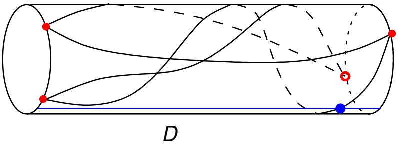

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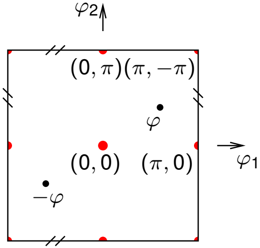
$$\mathcal{I}(D) = \text{parity of number of crossings of fiducial line}$$

Time reversal invariant bundles $(E, \mathbb{T}, \Theta)$

$\mathbb{T}$



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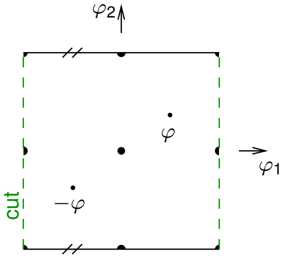
Towards another index

Consider the cut torus:



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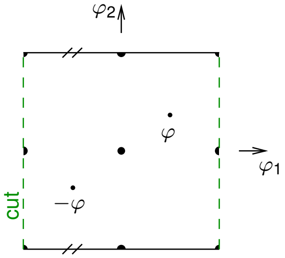

Lemma On the cut torus the frame bundle admits a section $\varphi \mapsto v(\varphi) \in F(E)_\varphi$ which is time-reversal invariant:

$$v(-\varphi) = (\Theta v(\varphi))\varepsilon$$

with ε the block diagonal matrix with blocks $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Idea: At a time reversal invariant point, that means ($N = 2$)

$$v_2 = \Theta v_1 \qquad v_1 = -\Theta v_2$$

Towards another index (cont.)

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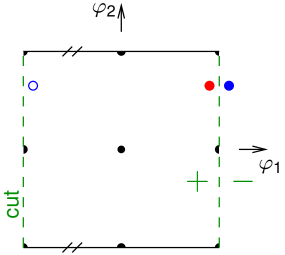

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$$v_+(\varphi_2) = v_-(\varphi_2)T(\varphi_2)\,, \qquad (\varphi_2 \in S^1)$$

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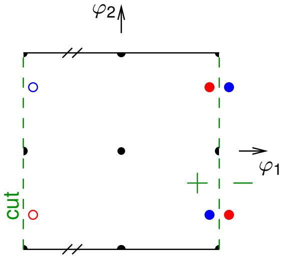

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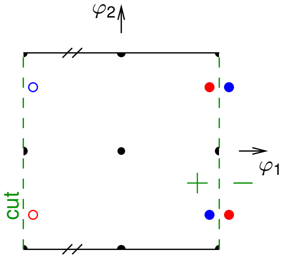

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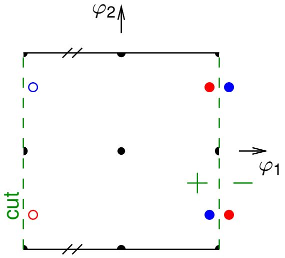

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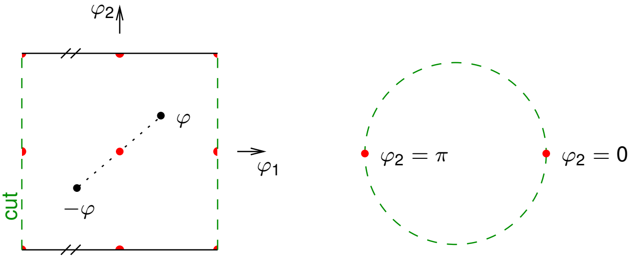
with $\Theta_0 = \varepsilon C$, (C complex conjugation on $\mathbb{C}^N$)

Towards another index (cont.)

We have

- torus $\varphi = (\varphi_1, \varphi_2) \in \mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^2$ with cut (figure)



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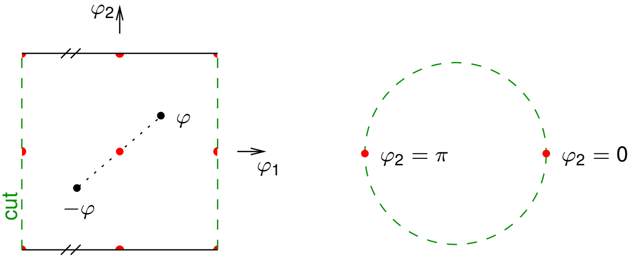

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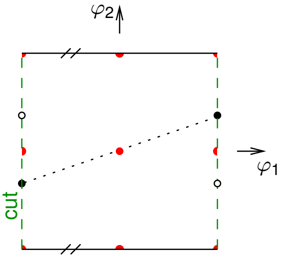

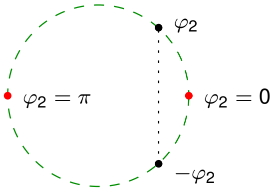

- a (compatible) section of the frame bundle of E
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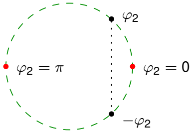
with $\Theta_0 : \mathbb{C}^N \to \mathbb{C}^N$ antilinear, $\Theta_0^2 = -1$

Time-reversal invariant bundles on the torus



- $\Theta_0 T(\varphi_2) = T^{-1}(-\varphi_2)\Theta_0$
- Only half the cut ($0 \leq \varphi_2 \leq \pi$) matters for $T(\varphi_2)$
- At time-reversal invariant points, $\varphi_2 = 0, \pi$,

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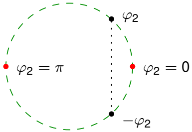

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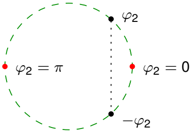

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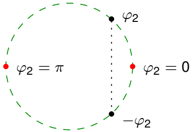
Phases $\lambda/|\lambda|$ pair up (degeneracy)

Time-reversal invariant bundles on the torus



- $\Theta_0 T(\varphi_2) = T^{-1}(-\varphi_2)\Theta_0$
- Only half the cut ($0 \leq \varphi_2 \leq \pi$) matters for $T(\varphi_2)$
- At time-reversal invariant points, $\varphi_2 = 0, \pi$,

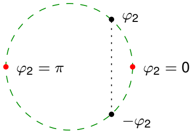
$$\Theta_0 T = T^{-1}\Theta_0$$

  Eigenvalues of T come in pairs $\lambda, \bar{\lambda}^{-1}$:

$$\Theta_0(T - \lambda) = T^{-1}(1 - \bar{\lambda}T)\Theta_0$$

  Phases $\lambda/|\lambda|$ pair up (degeneracy)
- For $0 \leq \varphi_2 \leq \pi$, phases $\lambda/|\lambda|$ form a rueda, *D*

Time-reversal invariant bundles on the torus



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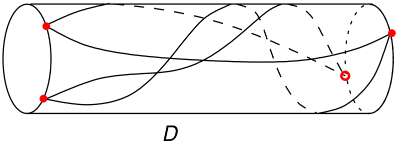

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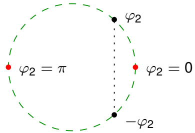

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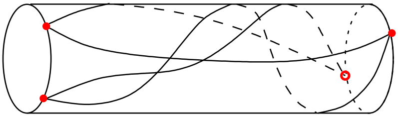

D

Definition (Index): $\mathcal{I}(E) := \mathcal{I}(T) := \mathcal{I}(D)$

Why are the $\mathbb{Z}_2$ indices equal?

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- Let $u(\varphi)$ be global frame as in the Fu-Kane index:

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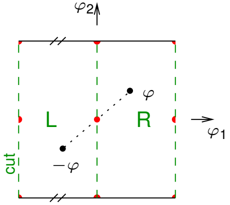
- Define frame $v(\phi)$

$$v(\varphi) = \begin{cases} u(\varphi)\ , & (\varphi \in L) \\ \Theta u(-\varphi)\varepsilon\ , & (\varphi \in R) \end{cases}$$



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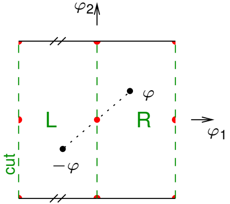

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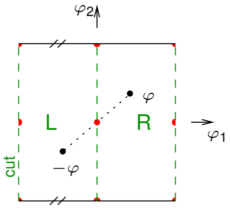

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- $W(\varphi_2) = T(\varphi_2)\varepsilon$. (crucial)

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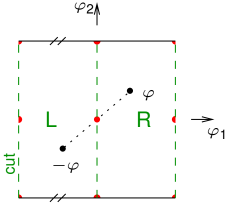

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$$\mathcal{I}(E) = \mathcal{I}(T_0)\mathcal{I}(T_\pi) \qquad \text{(ruedas)}$$

- $W(\varphi_2) = T(\varphi_2)\varepsilon$. Then $\widehat{\mathcal{I}}(W) = \mathcal{I}(T)$ and hence

$$\widehat{\mathcal{I}}(E) = \mathcal{I}(E)$$

Rueda de casino. Time 0′15″

Rueda de casino. Time 0′25″

Rueda de casino. Time $0'35''$

Rueda de casino. Time $0'44''$

Rueda de casino. Time $0'44.25''$

Rueda de casino. Time $0'44.50''$

Rueda de casino. Time $0'44.75''$

Rueda de casino. Time 0′45″

Rueda de casino. Time $0'45.25''$

Rueda de casino. Time 0′45.50″

Rueda de casino. Time 0′46″

Rueda de casino. Time 0′47″

Rueda de casino. Time $0'55''$

Rueda de casino. Time 1′16″

Rueda de casino. Time $3'23''$

Rules of the dance

Dancers

- start in pairs, anywhere
- end in pairs, anywhere (possibly elseways & elsewhere)
- are free in between
- must never step on center of the floor

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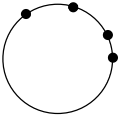
There are dances which can not be deformed into one another.

What is the index that tells the difference?

The index of a Rueda

A snapshot of the dance



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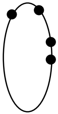
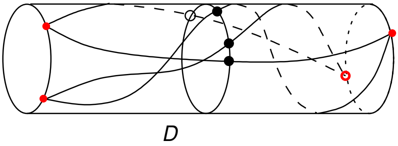


Dance D as a whole



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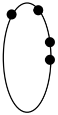

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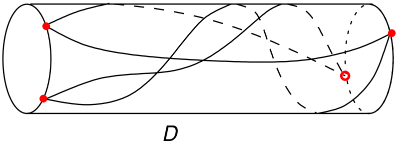

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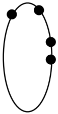

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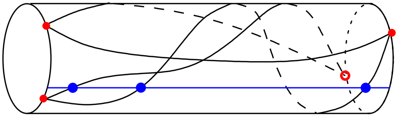

D

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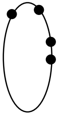

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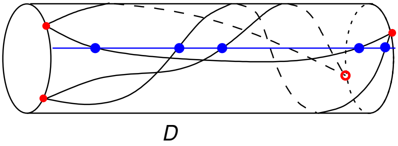

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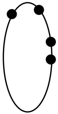
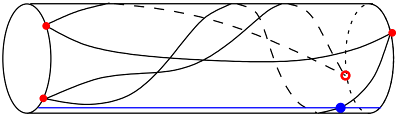


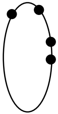
Dance *D* as a whole



D

The index of a Rueda

A snapshot of the dance



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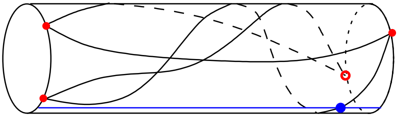

D

$\mathcal{I}(D) =$ parity of number of crossings of fiducial line

The $\mathbb{Z}_2$ index in the non-periodic case

Recall: Index without time-reversal symmetry based on index of pair of projections

$$\begin{aligned}&\mathrm{Ind}(P,Q) = \\ &\dim\{\psi \in \mathcal{H} \mid P\psi = \psi, Q\psi = 0\} - \dim\{\psi \in \mathcal{H} \mid Q\psi = \psi, P\psi = 0\} \\ &\qquad = \dim\ker(A-1) - \dim\ker(A+1), \qquad A = P - Q\end{aligned}$$

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$$= \dim \ker(A - 1) - \dim \ker(A + 1), \qquad A = P - Q$$

With time-reversal symmetry:

$$\mathcal{I} = (-1)^{\dim \ker(A-1)}$$

(cf. Atiyah; Schulz-Baldes; Katsura, Koma)

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In both cases, apply to $P = P_\mu$, $Q = UP_\mu U^*$.

An experiment: Amo et al.

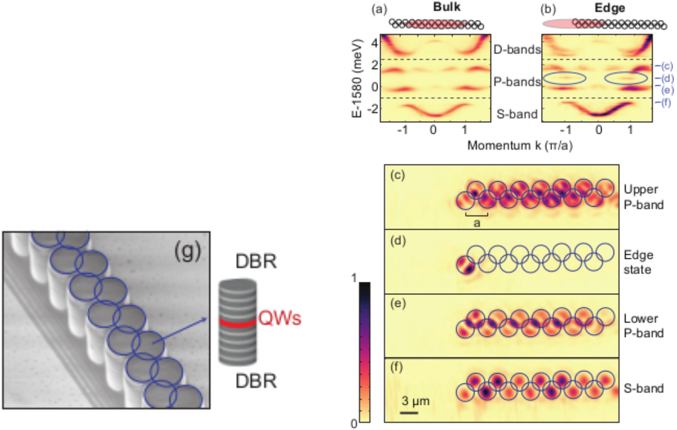
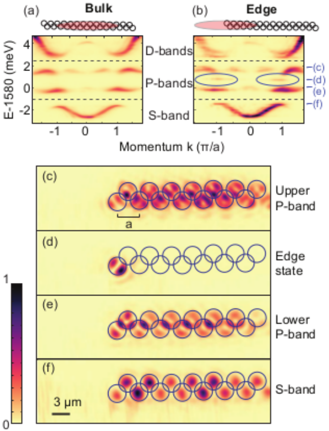


Figure: Zigzag chain of coupled micropillars and lasing modes

An experiment: Amo et al.



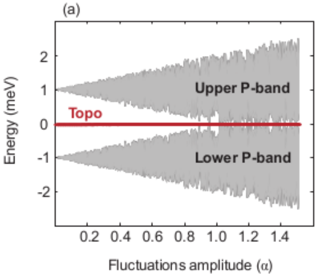
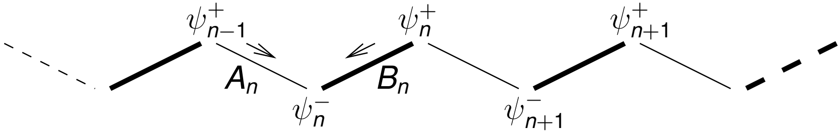


Figure: Lasing modes: bulk and edge

The Su-Schrieffer-Heeger model (1 dimensional)

Alternating chain with nearest neighbor hopping



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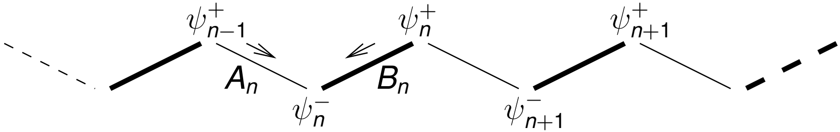

Hilbert space: sites arranged in dimers

$$\mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^N) \otimes \mathbb{C}^2 \ni \psi = \begin{pmatrix} \psi^+_n \\ \psi^-_n \end{pmatrix}_{n\in\mathbb{Z}}$$

Hamiltonian

$$H = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$$

with S, S^* acting on $\ell^2(\mathbb{Z}, \mathbb{C}^N)$ as

$$(S\psi^+)_n = A_n\psi^+_{n-1} + B_n\psi^+_n\,, \qquad (S^*\psi^-)_n = A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n$$

($A_n, B_n \in \mathrm{GL}(N)$ almost surely)

Chiral symmetry

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\{H, \Pi\} \equiv H\Pi + \Pi H = 0$$

hence

$$H\psi = \lambda\psi \quad \Longrightarrow \quad H(\Pi\psi) = -\lambda(\Pi\psi)$$

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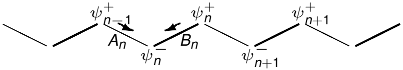


- Eigenvalue equation $H\psi = \lambda\psi$ is $S\psi^+ = \lambda\psi^-$, $S^*\psi^- = \lambda\psi^+$, i.e.

$$A_n\psi^+_{n-1} + B_n\psi^+_n = \lambda\psi^-_n\,, \qquad A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n = \lambda\psi^+_n$$

is one 2nd order difference equation, but two 1st order for $\lambda = 0$

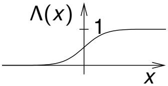
Bulk index

Let

$$\Sigma = \operatorname{sgn} H$$

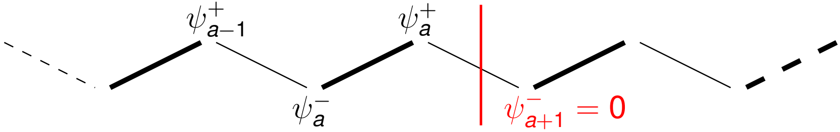
Definition. The Bulk index is

$$\mathcal{N} = \frac{1}{2} \operatorname{tr}(\Pi \Sigma [\Lambda, \Sigma])$$



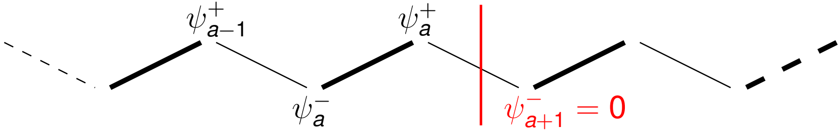
with $\Lambda = \Lambda(n)$ a switch function (cf. Prodan et al.)

Edge Hamiltonian and index



Edge Hamiltonian H_a defined by restriction to $n \leq a$ (Dirichlet boundary condition $\psi^-_{a+1} = 0$). Chiral symmetry preserved.

Edge Hamiltonian and index



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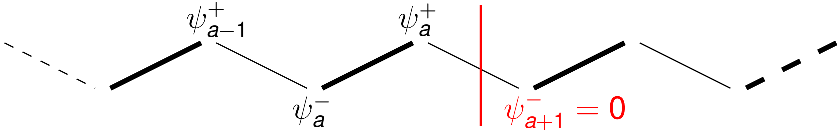

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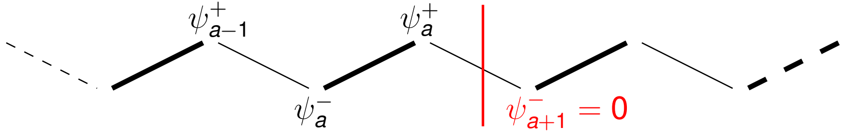
$$\mathcal{N}_a^\pm := \dim\{\psi \mid H_a\psi = 0, \Pi\psi = \pm\psi\}$$

Edge Hamiltonian and index



Edge Hamiltonian H_a defined by restriction to $n \leq a$ (Dirichlet boundary condition $\psi^-_{a+1} = 0$). Chiral symmetry preserved.

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$$\mathcal{N}^{\pm}_a := \dim\{\psi \mid H_a\psi = 0, \Pi\psi = \pm\psi\}$$

Definition. The Edge index is the spectral asymmetry

$$\mathcal{N}^{\sharp}_a := \mathcal{N}^{+}_a - \mathcal{N}^{-}_a$$

and can be shown to be independent of a. Call it $\mathcal{N}^{\sharp}$.

Bulk-edge duality

Theorem (G., Shapiro). Assume $\lambda = 0$ lies in a mobility gap. Then

$$\mathcal{N} = \mathcal{N}^{\sharp}$$

Bulk-edge duality: Remarks

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Remarks.

- Spectral gap case ($0 \notin \sigma_{\text{ess}}(H) \supset \sigma_{\text{ess}}(H_a)$)

$$H_a = \begin{pmatrix} 0 & S_a^* \\ S_a & 0 \end{pmatrix} \qquad \Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{N}_a^\sharp := \dim\ker S_a - \dim\ker S_a^* = \operatorname{ind} S_a \qquad \text{(Fredholm index)}$$

Bulk-edge duality by Schulz-Baldes. In mobility gap case, S_a is not Fredholm.

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- Periodic case

$$S = \int_{S^1}^{\oplus} S(k)$$

  Toeplitz index theorem:

$$\mathcal{N}^\sharp = -\text{Wind}(k \mapsto \det S(k))$$

Bulk-edge duality: Lyapunov exponents

Theorem (G., Shapiro). Assume $\lambda = 0$ lies in a mobility gap. Then

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Bulk-edge duality: Lyaponov exponents

Theorem (G., Shapiro). Assume $\lambda = 0$ lies in a mobility gap. Then

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Remark. Consider the dynamical system $A_n \psi^+_{n-1} + B_n \psi^+_n = 0$ with Lyaponov exponents

$$\gamma_1 \geq \dots \geq \gamma_N$$

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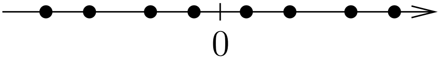
- at energy $\lambda \neq 0$ (simple spectrum)



  - Spectrum is simple because measure on transfer matrices is irreducible
  - so $\gamma = 0$ is not in the spectrum; localization follows

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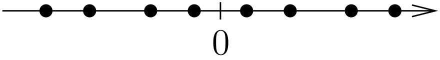

- At $\lambda = 0$ chains decouple: $\mathbb{C}^N \oplus 0$ and $0 \oplus \mathbb{C}^N$ are invariant subspaces

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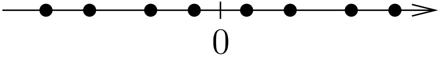
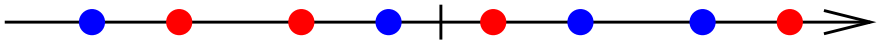
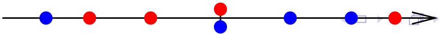


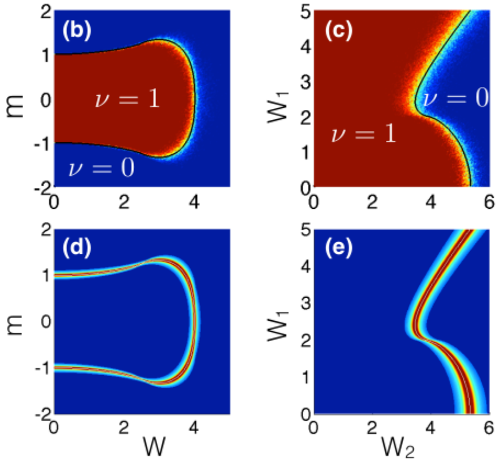
- of the upper ($+$) and lower ($-$) chains, at energy $\lambda = 0$



- at energy $\lambda = 0$ (phase boundary)



Some numerics



Left/right column: two parameterized chiral models ($N = 1$)
upper/lower row: index and Lyapunov exponent (from Prodan et al.)

Proof

Recall $\mathcal{N}_a = \mathrm{tr}(\Pi P_{0,a})$

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Proof of Theorem. On the Hilbert space $\mathcal{H}_a$ corresponding to $n \leq a$

$$\mathrm{tr}(\Pi \Lambda) = N\Big(\sum_{n \leq a} \Lambda(n)\Big)\, \mathrm{tr}_{\mathbb{C}^2} \Pi = 0$$

though $\|\Pi\Lambda\|_1 = \|\Lambda\|_1 \to \infty$, $(a \to +\infty)$

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So,

$$\mathrm{tr}(\Pi\Lambda) = \underbrace{\mathrm{tr}(\Pi\Lambda P_{0,a})}_{\to \mathcal{N}^\sharp} + \underbrace{\mathrm{tr}(\Pi\Lambda P_{+,a}) + \mathrm{tr}(\Pi\Lambda P_{-,a})}_{\to \mathrm{tr}(\Pi P_-[\Lambda, P_+]) + \mathrm{tr}(\Pi P_+[\Lambda, P_-]) = -\mathcal{N}}$$

q.e.d.

Floquet topological insulators

$H = H(t)$ (bulk) Hamiltonian in the plane with period T

$$H(t + T) = H(t)$$

(disorder allowed, no adiabatic setting)

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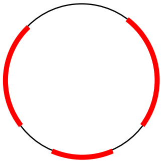
Assumption: Spectrum of $\widehat{U}$ has gaps:



$\operatorname{spec} \widehat{U} \subset S^1$

Bulk index

Special case first: $U(t)$ periodic, i.e.

$$\widehat{U} = 1$$

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$$\mathcal{N}_{\mathrm{B}} = \frac{1}{2}\int_0^T dt\,\mathrm{tr}\big(U^*\partial_t U\big[U^*[\Lambda_1, U], U^*[\Lambda_2, U]\big]\big)$$

with $U = U(t)$ and switches $\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$

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Remark. Extends the formula for the periodic case (Rudner et al.)

$$\mathcal{N}_{\mathrm{B}} = \frac{1}{8\pi^2} \int_0^T dt \int_{\mathbb{T}} d^2k \, \mathrm{tr}(U^* \partial_t U[U^* \partial_1 U, U^* \partial_2 U])$$

with $U = U(t, k)$ acting on the space of states of quasi-momentum $k = (k_1, k_2)$. Map U: 3-torus $\to$ unitary group $\mathcal{U}$; $\pi_3(\mathcal{U}) = \mathbb{Z}$.

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$H_{\mathrm{E}}(t)$ restriction of $H(t)$ to right half-space $x_1 > 0$

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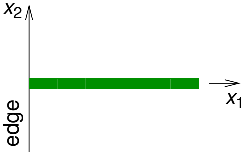
Remarks.

- The trace is well-defined



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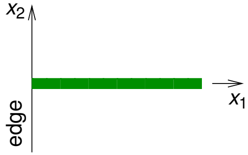

- $\mathcal{N}_{\mathrm{E}}$ is charge that crossed the line $x_2 = 0$ during a period.
- $\mathcal{N}_{\mathrm{E}}$ is independent of Λ_2 and an integer.

General case: Pair of Hamiltonians

$$\widehat{U} \neq 1$$

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$$H(t) = \begin{cases} H_1(t) & (0 < t < T) \\ -H_2(-t) & (-T < t < 0) \end{cases}$$

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Theorem (G., Tauber) $\mathcal{N} = \mathcal{N}_{\mathrm{E}}$

Duality in time and space

Let the interface Hamiltonian $H_{\mathrm{I}}(t)$ be a bulk Hamiltonian with

$$H_{\mathrm{I}}(t) = \begin{cases} H_1(t) \\ H_2(t) \end{cases} \quad \text{on states supported on large } \pm x_1$$

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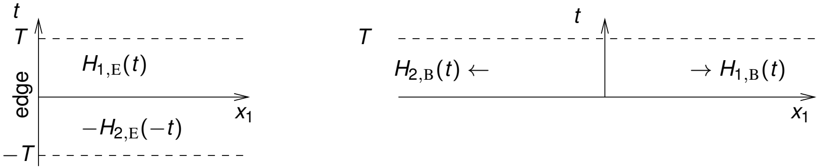
Interface index

$$\mathcal{N}_{\mathrm{I}} = \mathrm{tr}(\widehat{U}_\bullet^* \widehat{U}_{\mathrm{I}}[\Lambda_2, \widehat{U}_\bullet^* \widehat{U}_{\mathrm{I}}])$$



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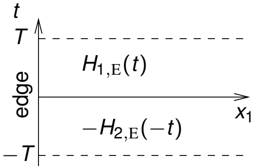
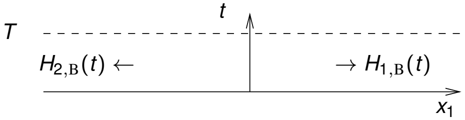


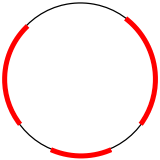


Theorem (G., Tauber) The indices for the two diagrams agree:

$$(\mathcal{N} =)\mathcal{N}_{\mathrm{E}} = \mathcal{N}_{\mathrm{I}}$$

Back to single Hamiltonian

$\widehat{U} \neq 1$



$\text{spec}\,\widehat{U} \subset S^1$

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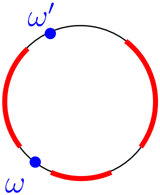

Let $\alpha \in \mathbb{R}$ and $\omega = \mathrm{e}^{\mathrm{i}\alpha}$. For $z \notin \omega\mathbb{R}_+$ (ray) define the branch

$$\log_\alpha z = \log|z| + \mathrm{i}\arg_\alpha z$$

by $\alpha - 2\pi < \arg_\alpha z < \alpha$.

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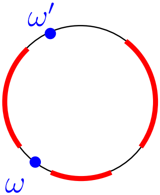

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Comparison Hamiltonian H_α: For $\omega \notin \mathrm{spec}\,\widehat{U}$ set

$$-\mathrm{i}H_\alpha T := \log_\alpha \widehat{U}$$

So,

- $\widehat{U}_\alpha = \widehat{U}$
- $U_{\alpha+2\pi}(t) = U_\alpha(t)\mathrm{e}^{2\pi\mathrm{i}t/T}$
- $\mathcal{N}_{\mathrm{B},\alpha+2\pi} = \mathcal{N}_{\mathrm{B},\alpha} =: \mathcal{N}_\omega$

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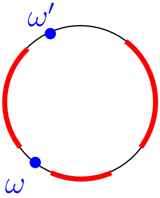

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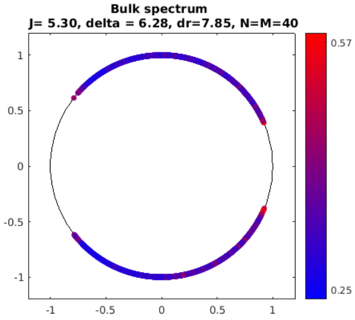
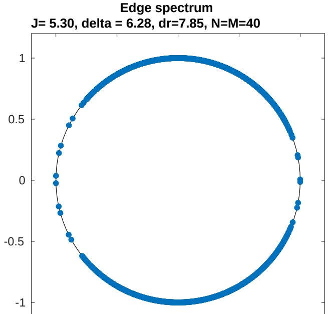
Theorem (Rudner et al.; G., Tauber) For ω, ω' in gaps

$$\mathcal{N}_{\omega'} - \mathcal{N}_\omega = i \operatorname{tr} P\big[[P, \Lambda_1], [P, \Lambda_2]\big]$$

where $P = P_{\omega,\omega'}$ is the spectral projection associated with $\operatorname{spec} \widehat{U}$ between ω, ω' (counter-clockwise)

Bulk and Edge spectrum



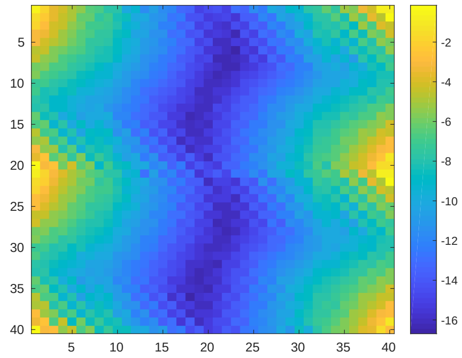


Computing the edge index

Edge index based $\mathcal{N}_{\mathrm{E},\alpha}$ based on the pair (H, H_α) (with $\alpha = \pi$)

$$\mathcal{N}_{\mathrm{E},\alpha} = \operatorname{tr} A \qquad A = \widehat{U}_{\mathrm{E}}^{*} \Lambda_2 \widehat{U}_{\mathrm{E}} - \widehat{U}_{\alpha,\mathrm{E}}^{*} \Lambda_2 \widehat{U}_{\alpha,\mathrm{E}}$$

The diagonal integral kernel $A(x,x)$ as $\log |A(x,x)|$



Boundary conditions:

- Vertical edges: Dirichlet
- Horizontal edges: Periodic

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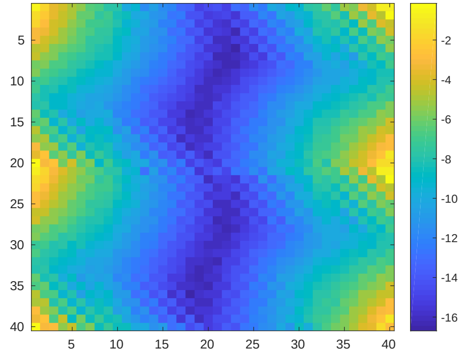

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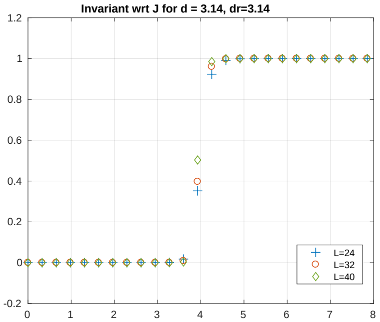
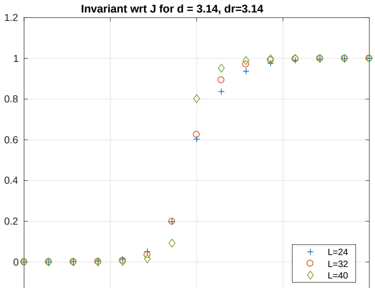
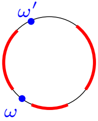
The transition





The anomalous phase

The spectrum of $\widehat{U}$ be fully localized (Rudner et al.): $\widehat{U}\psi_z = z\psi_z$, ($z$: eigenvalues $\in S^1$)



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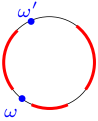

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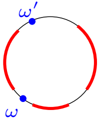

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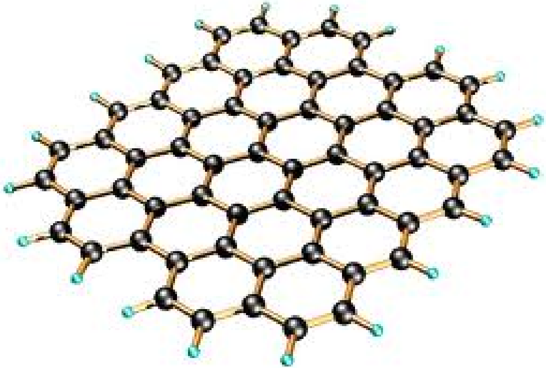
If H is time independent, then $\mathcal{M}(U) = 0$. So, for $(H_1(t), H_2(t)) = (H(t), H_\alpha)$ we have $\mathcal{N} = \mathcal{M}(U)$

Summary

Thank you for your attention!

Quantum Hall in graphene (cf. talk by S. Becker)

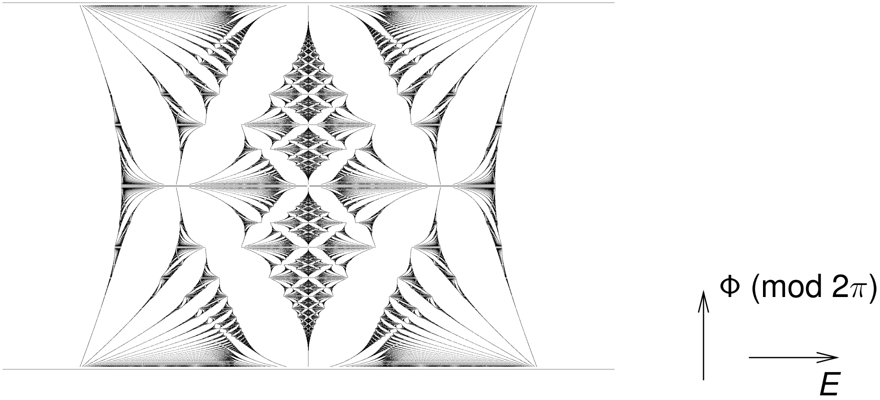


An application: Quantum Hall in graphene

Hamiltonian: Nearest neighbor hopping with flux Φ per plaquette.

An application: Quantum Hall in graphene

Spectrum in black



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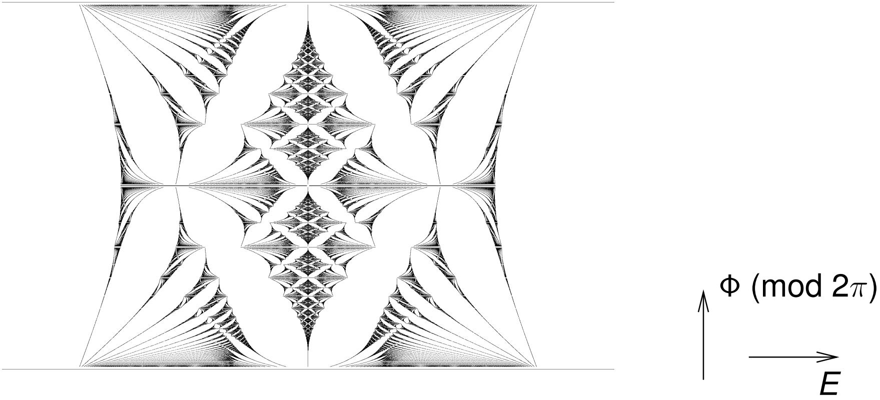

What is the Hall conductance (Chern number) in any white point?

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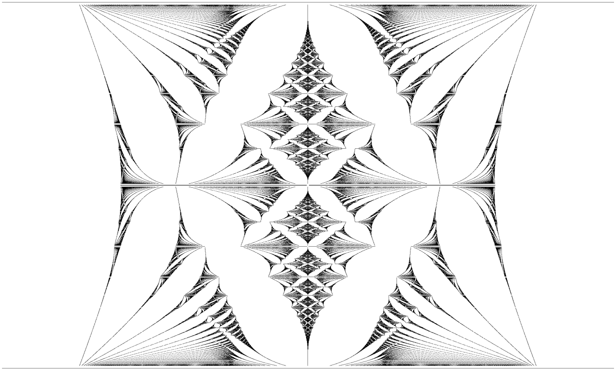

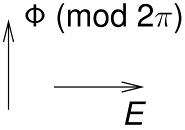

Answer: Edge approach, method by Schulz-Baldes et al.

The colors of graphene

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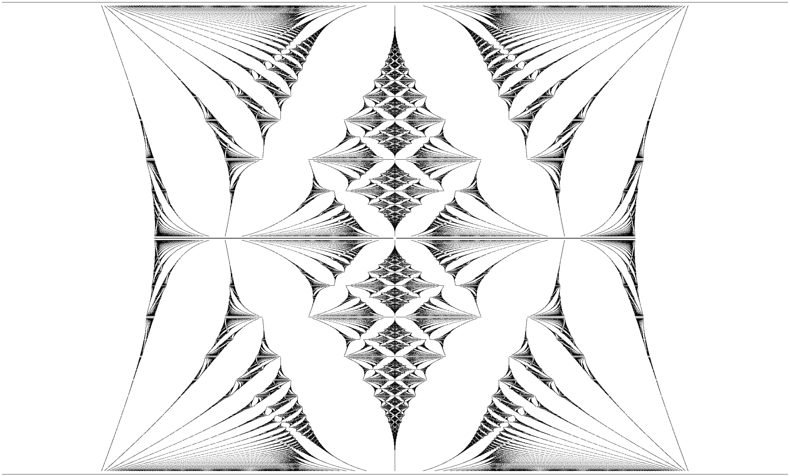

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